CHICAGO JOURNAL OF THEORETICAL COMPUTER SCIENCE
http://cjtcs.cs.uchicago.edu/

Simpler semidefinite programs for completely bounded norms

John Watrous*

July 11, 2013

Abstract: The completely bounded trace and spectral norms, for finite-dimensional spaces, are known to be efficiently expressible by semidefinite programs (J. Watrous, *Theory of Computing* 5: 11, 2009). This paper presents two new, and arguably simpler, semidefinite programming formulations of these norms.

1 Introduction and preliminary discussion

In the theory of quantum information, *quantum states* are represented by density operators acting on finite-dimensional complex vector spaces, while *quantum channels* are represented by linear mappings that transform one density operator into another [8, 10]. Various concepts connected with mappings of this form, meaning ones that map linear operators to linear operators (or, equivalently, that map matrices to matrices), are important in the study of quantum information for this and other reasons. Linear mappings of this form are also important in the study of operator algebras [11].

This paper is concerned specifically with the *completely bounded trace and spectral norms*, defined for linear mappings of the form just described. It is intended as a follow-up paper to [14], which demonstrated that these norms can be efficiently expressed and computed through the use of semidefinite programming. Two new semidefinite programming formulations of these norms will be presented. The new semidefinite programs themselves are comparable to those of [14] with respect to size, but they are more directly connected with the norms they represent and the analysis of their correctness is simpler and more intuitive.

*Supported by Canada's NSERC and the Canadian Institute for Advanced Research

Key words and phrases: Completely bounded norm, semidefinite programming.

A further discussion of the completely bounded trace and spectral norms can be found in [14]. That discussion will not be repeated here—instead, we will proceed directly to the technical content of the paper, beginning with a short summary of the notation and basic concepts that are to be assumed.

Linear algebra basics

For a complex vector space of the form $\mathcal{X} = \mathbb{C}^n$ and vectors $u = (\alpha_1, \ldots, \alpha_n)$ and $v = (\beta_1, \ldots, \beta_n)$ in $\mathcal{X}$, we define the inner product

$$\langle u, v \rangle = \sum_{j=1}^{n} \overline{\alpha_j} \beta_j$$

as well as the Euclidean norm

$$\| u \| = \sqrt{\langle u, u \rangle}.$$

For each $j \in \{1, \ldots, n\}$, the vector $e_j \in \mathcal{X}$ is defined to be the vector having a 1 in entry j and 0 for all other entries.

Given two complex vector spaces $\mathcal{X} = \mathbb{C}^n$ and $\mathcal{Y} = \mathbb{C}^m$, we denote the space of all linear mappings (or *operators*) of the form $A : \mathcal{X} \to \mathcal{Y}$ as $\mathrm{L}(\mathcal{X}, \mathcal{Y})$, and identify this space with the collection of all $m \times n$ complex matrices in the usual way. For each pair of indices (i, j) we write $E_{i,j}$ to denote the operator whose matrix representation has a 1 in entry (i, j) and zeroes in all other entries. The notation $\mathrm{L}(\mathcal{X})$ is shorthand for $\mathrm{L}(\mathcal{X}, \mathcal{X})$, and the identity operator on $\mathcal{X}$, which is an element of $\mathrm{L}(\mathcal{X})$, is denoted $\mathbb{1}_{\mathcal{X}}$. (The notation $\mathbb{1}$ is sometimes used in place of $\mathbb{1}_{\mathcal{X}}$ when it is clear that we are referring to the identity operator on $\mathcal{X}$.)

For each operator $A \in \mathrm{L}(\mathcal{X}, \mathcal{Y})$, one defines $A^* \in \mathrm{L}(\mathcal{Y}, \mathcal{X})$ to be the unique operator satisfying $\langle v, Au \rangle = \langle A^* v, u \rangle$ for all $u \in \mathcal{X}$ and $v \in \mathcal{Y}$. As a matrix, A^* is obtained by taking the conjugate transpose of the matrix associated with A. An inner product on $\mathrm{L}(\mathcal{X}, \mathcal{Y})$ is defined as

$$\langle A, B \rangle = \operatorname{Tr}(A^* B)$$

for all $A, B \in \mathrm{L}(\mathcal{X}, \mathcal{Y})$. By identifying a given vector $u \in \mathcal{X}$ with the linear mapping $\alpha \mapsto \alpha u$, which is an element of $\mathrm{L}(\mathbb{C}, \mathcal{X})$, the mapping $u^* \in \mathrm{L}(\mathcal{X}, \mathbb{C})$ is defined. More explicitly, u^* is the mapping that satisfies $u^* v = \langle u, v \rangle$ for all $v \in \mathcal{X}$.

An operator $X \in \mathrm{L}(\mathcal{X})$ is *Hermitian* if $X = X^*$, and the set of such operators is denoted $\mathrm{Herm}(\mathcal{X})$. An operator $X \in \mathrm{L}(\mathcal{X})$ is *positive semidefinite* if it is Hermitian and all of its eigenvalues are nonnegative. The set of such operators is denoted $\mathrm{Pos}(\mathcal{X})$. The notation $X \geq 0$ also indicates that X is positive semidefinite, and more generally the notations $X \leq Y$ and $Y \geq X$ indicate that $Y - X \geq 0$ for Hermitian operators X and Y. For every $P \in \mathrm{Pos}(\mathcal{X})$, one writes $\sqrt{P}$ to denote the unique positive semidefinite operator $Q \in \mathrm{Pos}(\mathcal{X})$ satisfying $Q^2 = P$. An operator $X \in \mathrm{L}(\mathcal{X})$ is *positive definite* if it is both positive semidefinite and invertible. Equivalently, X is positive definite if it is Hermitian and all of its eigenvalues are positive. The set of such operators is denoted $\mathrm{Pd}(\mathcal{X})$. The notation $X > 0$ also indicates that X is positive definite, and the notations $X < Y$ and $Y > X$ indicate that $Y - X > 0$ for Hermitian operators X and Y. An operator $\rho \in \mathrm{L}(\mathcal{X})$ is a *density operator* if it is both positive semidefinite and has trace equal to 1, and the set of such operators is denoted $\mathrm{D}(\mathcal{X})$. Finally, an operator $U \in \mathrm{L}(\mathcal{X})$ is *unitary* if $U^* U = \mathbb{1}_{\mathcal{X}}$, and the set of such operators is denoted $\mathrm{U}(\mathcal{X})$.

For $\mathcal{X} = \mathbb{C}^n$ and $\mathcal{Y} = \mathbb{C}^m$, the space of all linear mappings of the form $\Phi : \mathrm{L}(\mathcal{X}) \to \mathrm{L}(\mathcal{Y})$ is denoted $\mathrm{T}(\mathcal{X},\mathcal{Y})$. For each $\Phi \in \mathrm{T}(\mathcal{X},\mathcal{Y})$, the mapping $\Phi^* \in \mathrm{T}(\mathcal{Y},\mathcal{X})$ is the unique mapping for which the equation

$$\langle Y, \Phi(X)\rangle = \langle \Phi^*(Y), X\rangle$$

holds for all $X \in \mathrm{L}(\mathcal{X})$ and $Y \in \mathrm{L}(\mathcal{Y})$. A mapping $\Phi \in \mathrm{T}(\mathcal{X},\mathcal{Y})$ is *Hermiticity preserving* if it holds that $\Phi(X) \in \mathrm{Herm}(\mathcal{Y})$ for all choices of $X \in \mathrm{Herm}(\mathcal{X})$, and is *positive* if it holds that $\Phi(X) \in \mathrm{Pos}(\mathcal{Y})$ for all choices of $X \in \mathrm{Pos}(\mathcal{X})$. We denote by $\mathbb{1}_{\mathrm{L}(\mathcal{X})}$ the identity mapping on $\mathrm{L}(\mathcal{X})$.

Direct sums and tensor products are defined in the standard way for spaces of the form $\mathcal{X} = \mathbb{C}^n$, $\mathcal{Y} = \mathbb{C}^m$, and so on, as are tensor products of operators and mappings of the forms discussed above.

Norms and fidelity

For $\mathcal{X} = \mathbb{C}^n$, $\mathcal{Y} = \mathbb{C}^m$, and any operator $A \in \mathrm{L}(\mathcal{X},\mathcal{Y})$, one defines the *trace norm*, *Frobenius norm*, and *spectral norm* of A as

$$\|A\|_1 = \operatorname{Tr}\sqrt{A^*A}, \quad \|A\|_2 = \sqrt{\langle A, A\rangle}, \quad \text{and} \quad \|A\|_\infty = \max\big\{\|Au\| : u \in \mathcal{X},\, \|u\| \le 1\big\},$$

respectively. These norms correspond precisely to the 1-norm, 2-norm, and ∞-norm of the vector of singular values of A. All three of these norms are *unitarily invariant*, meaning that

$$\|UAV\|_1 = \|A\|_1, \qquad \|UAV\|_2 = \|A\|_2, \qquad \text{and} \qquad \|UAV\|_\infty = \|A\|_\infty$$

for every operator $A \in \mathrm{L}(\mathcal{X},\mathcal{Y})$ and every choice of unitary operators $U \in \mathrm{U}(\mathcal{Y})$ and $V \in \mathrm{U}(\mathcal{X})$. For every operator $A \in \mathrm{L}(\mathcal{X},\mathcal{Y})$ it holds that

$$\|A\|_\infty \le \|A\|_2 \le \|A\|_1.$$

The trace and spectral norms are dual to one another, meaning

$$\|A\|_1 = \max\big\{|\langle B, A\rangle| : \|B\|_\infty \le 1\big\},$$
$$\|A\|_\infty = \max\big\{|\langle B, A\rangle| : \|B\|_1 \le 1\big\},$$

for all $A \in \mathrm{L}(\mathcal{X},\mathcal{Y})$, and with B ranging over operators within the same space.

For each $\Phi \in \mathrm{T}(\mathcal{X},\mathcal{Y})$, one defines the *induced* trace and spectral norms as

$$\|\Phi\|_1 = \max\big\{\|\Phi(X)\|_1 : X \in \mathrm{L}(\mathcal{X}),\ \|X\|_1 \le 1\big\},$$
$$\|\Phi\|_\infty = \max\big\{\|\Phi(X)\|_\infty : X \in \mathrm{L}(\mathcal{X}),\ \|X\|_\infty \le 1\big\},$$

as well as *completely bounded* variants of these norms:

$$|||\Phi|||_1 = \sup_{k\ge 1}\big\|\Phi \otimes \mathbb{1}_{\mathrm{L}(\mathbb{C}^k)}\big\|_1 = \big\|\Phi \otimes \mathbb{1}_{\mathrm{L}(\mathcal{X})}\big\|_1,$$
$$|||\Phi|||_\infty = \sup_{k\ge 1}\big\|\Phi \otimes \mathbb{1}_{\mathrm{L}(\mathbb{C}^k)}\big\|_\infty = \big\|\Phi \otimes \mathbb{1}_{\mathrm{L}(\mathcal{Y})}\big\|_\infty.$$

By the duality of the trace and spectral norms, it holds that

$$|||\Phi|||_1 = |||\Phi^*|||_\infty \tag{1.1}$$

for every mapping $\Phi \in \mathrm{T}(\mathcal{X},\mathcal{Y})$. In the subsequent sections of the paper, our focus will be on semidefinite programming formulations of the completely bounded trace norm $|||\cdot|||_1$; interested readers may directly adapt these formulations to ones for the complete bounded spectral norm by means of the relationship (1.1). As every operator X having trace norm bounded by 1 can be written as a convex combination of rank 1 operators taking the form uv^* for u and v being unit vectors, it follows from convexity that

$$|||\Phi|||_1 = \max\left\{ \left\| (\Phi \otimes \mathbb{1}_{\mathrm{L}(\mathcal{X})})(uv^*) \right\|_1 : u,v \in \mathcal{X}\otimes\mathcal{X},\, \|u\| = \|v\| = 1 \right\}. \tag{1.2}$$

Finally, for any two positive semidefinite operators $P,Q \in \mathrm{Pos}(\mathcal{X})$, one defines the *fidelity* between P and Q as

$$\mathrm{F}(P,Q) = \left\| \sqrt{P}\sqrt{Q} \right\|_1. \tag{1.3}$$

For $u,v \in \mathcal{X}\otimes\mathcal{Y}$ being any choice of vectors, it holds that

$$\mathrm{F}\big(\mathrm{Tr}_{\mathcal{Y}}(uu^*),\mathrm{Tr}_{\mathcal{Y}}(vv^*)\big) = \|\mathrm{Tr}_{\mathcal{X}}(vu^*)\|_1, \tag{1.4}$$

for $\mathrm{Tr}_{\mathcal{X}}$ and $\mathrm{Tr}_{\mathcal{Y}}$ denoting the partial traces on $\mathcal{X}$ and $\mathcal{Y}$ respectively. These are the unique linear mappings $\mathrm{Tr}_{\mathcal{X}} \in \mathrm{T}(\mathcal{X}\otimes\mathcal{Y},\mathcal{Y})$ and $\mathrm{Tr}_{\mathcal{Y}} \in \mathrm{T}(\mathcal{X}\otimes\mathcal{Y},\mathcal{X})$ satisfying $\mathrm{Tr}_{\mathcal{X}}(X\otimes Y) = \mathrm{Tr}(X)Y$ and $\mathrm{Tr}_{\mathcal{Y}}(X\otimes Y) = \mathrm{Tr}(Y)X$ for all $X \in \mathrm{L}(\mathcal{X})$ and $Y \in \mathrm{L}(\mathcal{Y})$. Proofs of the identity (1.4) may be found in [12] and [13].

Semidefinite programming

A *semidefinite program*[1] is specified by a triple (Ξ,C,D), where

1. $\Xi \in \mathrm{T}(\mathcal{X},\mathcal{Y})$ is a Hermiticity-preserving linear map, and
2. $C \in \mathrm{Herm}(\mathcal{X})$ and $D \in \mathrm{Herm}(\mathcal{Y})$ are Hermitian operators,

for $\mathcal{X} = \mathbb{C}^n$ and $\mathcal{Y} = \mathbb{C}^m$ denoting spaces as before. We associate with the triple (Ξ,C,D) two optimization problems, called the *primal* and *dual* problems, as follows:

Primal problem		Dual problem	
maximize:	$\langle C,X\rangle$	minimize:	$\langle D,Y\rangle$
subject to:	$\Xi(X) = D,$	subject to:	$\Xi^*(Y) \geq C,$
	$X \in \mathrm{Pos}(\mathcal{X}).$		$Y \in \mathrm{Herm}(\mathcal{Y}).$

An operator $X \in \mathrm{Pos}(\mathcal{X})$ satisfying $\Xi(X) = D$ is said to be *primal feasible*, and an operator $Y \in \mathrm{Herm}(\mathcal{Y})$ satisfying $\Xi^*(Y) \geq C$ is said to be *dual feasible*. We let $\mathcal{P}$ and $\mathcal{D}$ denote the sets of primal and dual feasible operators , respectively:

$$\mathcal{P} = \big\{X \in \mathrm{Pos}(\mathcal{X}) : \Xi(X) = D\big\} \qquad \text{and} \qquad \mathcal{D} = \big\{Y \in \mathrm{Herm}(\mathcal{Y}) : \Xi^*(Y) \geq C\big\}.$$

[1] It should be noted that the above definition differs slightly from the one in [14], where the equality constraint $\Xi(X) = D$ appears instead as an inequality constraint $\Xi(X) \leq D$, and (correspondingly) the dual condition $Y \in \mathrm{Herm}(\mathcal{Y})$ appears as $Y \in \mathrm{Pos}(\mathcal{Y})$. The two forms can easily be converted back and forth, but the one above is more convenient for the purposes of this paper.

The linear functions $X \mapsto \langle C, X\rangle$ and $Y \mapsto \langle D, Y\rangle$ are referred to as the primal and dual *objective functions*, which take real number values for all choices of $X \in \mathcal{P}$ and $Y \in \mathcal{D}$ (or, more generally, for all choices of $X \in \operatorname{Herm}(\mathcal{X})$ and $Y \in \operatorname{Herm}(\mathcal{Y})$). The *primal optimum* and *dual optimum* are defined as

$$\alpha = \sup_{X \in \mathcal{P}} \langle C, X\rangle \qquad \text{and} \qquad \beta = \inf_{Y \in \mathcal{D}} \langle D, Y\rangle,$$

respectively. The values α and β may be finite or infinite, and by convention we define $\alpha = -\infty$ if $\mathcal{P} = \varnothing$ and $\beta = \infty$ if $\mathcal{D} = \varnothing$. If an operator $X \in \mathcal{P}$ satisfies $\langle C, X\rangle = \alpha$ we say that X is an *optimal primal solution*, or that X achieves the primal optimum. Likewise, if $Y \in \mathcal{D}$ satisfies $\langle D, Y\rangle = \beta$ we say that Y is an *optimal dual solution*, or that Y achieves the dual optimum.

For every semidefinite program it holds that $\alpha \leq \beta$, which is a fact known as *weak duality*. The condition $\alpha = \beta$, known as *strong duality*, may fail to hold for some semidefinite programs—but, for a wide range of semidefinite programs that arise in practice, strong duality does hold. The following theorem provides a condition (in both a primal and dual form) that implies strong duality.

Theorem 1.1 (Slater's theorem for semidefinite programs)**.** *The following implications hold for every semidefinite program* (Ξ, C, D).

1. *If $\mathcal{P} \neq \varnothing$ and there exists a Hermitian operator Y for which $\Xi^*(Y) > C$, then $\alpha = \beta$ and there exists a primal feasible operator $X \in \mathcal{P}$ for which $\langle C, X\rangle = \alpha$.*
2. *If $\mathcal{D} \neq \varnothing$ and there exists a positive definite operator $X > 0$ for which $\Xi(X) = D$, then $\alpha = \beta$ and there exists a dual feasible operator $Y \in \mathcal{D}$ for which $\langle D, Y\rangle = \beta$.*

The condition that some operator $X > 0$ satisfies $\Xi(X) = D$ is called *strict primal feasibility*, while the condition that some operator $Y \in \operatorname{Herm}(\mathcal{Y})$ satisfies $\Xi^*(Y) > C$ is called *strict dual feasibility*; in both cases, the "strictness" concerns the positive semidefinite ordering.

2 A semidefinite program for the maximum output fidelity

The first semidefinite programming formulation of the completely bounded trace norm to be presented in this paper is based on a characterization of the completely bounded trace norm in terms of the fidelity function, together with a simple semidefinite program for the fidelity function itself.

The same characterization of the completely bounded trace norm in terms of the fidelity function was used by Ben-Aroya and Ta-Shma [2] to prove that the completely bounded trace norm can be efficiently approximated using convex programming. The formulation that follows may be viewed as a semidefinite programming representation of their method.

2.1 A semidefinite program for the fidelity function

We will begin by presenting a semidefinite programming characterization of the fidelity $\mathrm{F}(P, Q)$ between two positive semidefinite operators $P, Q \in \operatorname{Pos}(\mathcal{X})$, for $\mathcal{X} = \mathbb{C}^n$. The same semidefinite programming characterization of the fidelity was independently discovered by Nathan Killoran [7].

The semidefinite program is given by the triple (Ξ, C, D), where $\Xi : \mathrm{L}(\mathcal{X} \oplus \mathcal{X}) \to \mathrm{L}(\mathcal{X} \oplus \mathcal{X})$ is defined as

$$\Xi \begin{pmatrix} W_0 & \cdot \\ \cdot & W_1 \end{pmatrix} = \begin{pmatrix} W_0 & 0 \\ 0 & W_1 \end{pmatrix}$$

for all $W_0, W_1 \in \mathrm{L}(\mathcal{X})$ (and where the dots represent operators upon which Ξ does not depend), and $C, D \in \mathrm{Herm}(\mathcal{X} \oplus \mathcal{X})$ are defined as

$$C = \frac{1}{2}\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \qquad \text{and} \qquad D = \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix}.$$

The primal and dual problems associated with this semidefinite program may, after some simplifications, be expressed as follows:

<u>Primal problem</u>

$$\begin{aligned} \text{maximize:} \quad & \frac{1}{2}\mathrm{Tr}(X) + \frac{1}{2}\mathrm{Tr}(X^*) \\ \text{subject to:} \quad & \begin{pmatrix} P & X \\ X^* & Q \end{pmatrix} \geq 0 \\ & X \in \mathrm{L}(\mathcal{X}). \end{aligned}$$

<u>Dual problem</u>

$$\begin{aligned} \text{minimize:} \quad & \frac{1}{2}\langle P, Y\rangle + \frac{1}{2}\langle Q, Z\rangle \\ \text{subject to:} \quad & \begin{pmatrix} Y & -\mathbb{1} \\ -\mathbb{1} & Z \end{pmatrix} \geq 0 \\ & Y, Z \in \mathrm{Herm}(\mathcal{X}). \end{aligned}$$

For every choice of Hermitian operators $Y, Z \in \mathrm{Herm}(\mathcal{X})$, it holds that

$$\begin{pmatrix} Y & -\mathbb{1} \\ -\mathbb{1} & Z \end{pmatrix} \geq 0$$

if and only if $Y, Z \in \mathrm{Pd}(\mathcal{X})$ and $Z \geq Y^{-1}$. As Q is positive semidefinite, implying that $\langle Q, Z\rangle \geq \langle Q, Y^{-1}\rangle$ when $Z \geq Y^{-1}$, it follows that the dual problem can be further simplified as follows:

<u>Dual problem (simplified)</u>

$$\begin{aligned} \text{minimize:} \quad & \frac{1}{2}\langle P, Y\rangle + \frac{1}{2}\langle Q, Y^{-1}\rangle \\ \text{subject to:} \quad & Y \in \mathrm{Pd}(\mathcal{X}). \end{aligned}$$

Strong duality

Strong duality for the semidefinite program (Ξ, C, D) may be verified through an application of Slater's theorem, using the fact that the primal problem is feasible and the dual problem is strictly feasible. In particular, the operator

$$\begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix}$$

is primal feasible, which implies that $\mathcal{P} \neq \varnothing$. For the dual problem (as defined formally by the triple (Ξ, C, D), before any simplifications), the operator

$$\begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}$$

is strictly feasible, as

$$\Xi^*\begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} > \frac{1}{2}\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}.$$

By Slater's theorem, we have strong duality, and moreover the primal optimum is achieved by some choice of a primal feasible operator.

It so happens that strict primal feasibility may fail to hold: if either of P or Q is not positive definite, it cannot hold that

$$\begin{pmatrix} P & X \\ X^* & Q \end{pmatrix} > 0.$$

One cannot conclude from this fact that the optimal dual value will not be achieved—but indeed this is the case for some choices of P and Q. If P and Q are positive definite, however, then strict primal feasibility does hold, and the existence of an optimal dual solution follows from Slater's theorem.

Optimal value

One may prove that the optimal value of the semidefinite program described above is equal to $\mathrm{F}(P,Q)$ by making use of the following fact (stated as Theorem IX.5.9 in [3]).

Lemma 2.1. *Let $P,Q \in \mathrm{Pos}(\mathbb{C}^n)$ be positive semidefinite operators and let $X \in \mathrm{L}(\mathbb{C}^n)$ be any operator. It holds that*

$$\begin{pmatrix} P & X \\ X^* & Q \end{pmatrix} \in \mathrm{Pos}(\mathbb{C}^n \oplus \mathbb{C}^n) \tag{2.1}$$

if and only if $X = \sqrt{P}K\sqrt{Q}$ for $K \in \mathrm{L}(\mathbb{C}^n)$ satisfying $\|K\|_\infty \leq 1$.

It follows from this lemma that for feasible solutions to the primal problem, the variable $X \in \mathrm{L}(\mathcal{X})$ is free to range over those operators given by $\sqrt{P}K\sqrt{Q}$ for $K \in \mathrm{L}(\mathcal{X})$ satisfying $\|K\|_\infty \leq 1$. The primal optimum is therefore given by

$$\begin{aligned}\sup_K \left(\frac{1}{2}\mathrm{Tr}\left(\sqrt{P}K\sqrt{Q}\right) + \frac{1}{2}\mathrm{Tr}\left(\sqrt{Q}K^*\sqrt{P}\right)\right) &= \sup_K \Re\left(\mathrm{Tr}\left(\sqrt{Q}K^*\sqrt{P}\right)\right) \\ &= \sup_K \left|\left(\mathrm{Tr}\left(\sqrt{Q}K^*\sqrt{P}\right)\right)\right| = \sup_K \left|\left\langle K, \sqrt{P}\sqrt{Q}\right\rangle\right| = \left\|\sqrt{P}\sqrt{Q}\right\|_1 = \mathrm{F}(P,Q),\end{aligned}$$

where each supremum is over the set $\{K \in \mathrm{L}(\mathcal{X}) : \|K\|_\infty \leq 1\}$.

By strong duality, the dual optimum is also equal to $\mathrm{F}(P,Q)$. An alternate way to prove this fact uses the following theorem of Alberti.

Theorem 2.2 (Alberti)**.** *Let $\mathcal{X} = \mathbb{C}^n$ and let $P,Q \in \mathrm{Pos}(\mathcal{X})$ be positive semidefinite operators. It holds that*

$$(\mathrm{F}(P,Q))^2 = \inf_{Y \in \mathrm{Pd}(\mathcal{X})} \langle P, Y\rangle \langle Q, Y^{-1}\rangle.$$

To see that Alberti's theorem implies that the dual optimum of the semidefinite program is equal to $\mathrm{F}(P,Q)$, note first that the arithmetic-geometric mean inequality implies that

$$\frac{1}{2}\langle P,Y\rangle + \frac{1}{2}\langle Q,Y^{-1}\rangle \geq \sqrt{\langle P,Y\rangle\langle Q,Y^{-1}\rangle}$$

for every $Y \in \mathrm{Pd}(\mathcal{X})$, with equality if and only if $\langle P,Y\rangle = \langle Q,Y^{-1}\rangle$. It follows that

$$\frac{1}{2}\langle P,Y\rangle + \frac{1}{2}\langle Q,Y^{-1}\rangle \geq \mathrm{F}(P,Q)$$

for every $Y \in \mathrm{Pd}(\mathcal{X})$. Moreover, for an arbitrary choice of $Y \in \mathrm{Pd}(\mathcal{X})$, one may choose $\lambda > 0$ so that

$$\langle P,\lambda Y\rangle = \langle Q,(\lambda Y)^{-1}\rangle$$

and therefore

$$\frac{1}{2}\langle P,\lambda Y\rangle + \frac{1}{2}\langle Q,(\lambda Y)^{-1}\rangle = \sqrt{\langle P,\lambda Y\rangle\langle Q,(\lambda Y)^{-1}\rangle} = \sqrt{\langle P,Y\rangle\langle Q,Y^{-1}\rangle}.$$

Thus, the dual optimum is given by

$$\inf_{Y\in\mathrm{Pd}(\mathcal{X})}\left(\frac{1}{2}\langle P,Y\rangle + \frac{1}{2}\langle Q,Y^{-1}\rangle\right) = \mathrm{F}(P,Q).$$

By reversing this argument, an alternate proof of Alberti's theorem based on semidefinite programming duality is obtained. A similar observation was made in [14] based on a different semidefinite programming formulation of the fidelity.

2.2 Maximum output fidelity characterization of the completely bounded trace norm

Next, we recall a known characterization of the completely bounded trace norm in terms of the fidelity function, which makes use of the following definition.

Definition 2.3. Let $\mathcal{X} = \mathbb{C}^n$ and $\mathcal{Z} = \mathbb{C}^k$, and let $\Psi_0, \Psi_1 \in \mathrm{T}(\mathcal{X},\mathcal{Z})$ be positive maps. The *maximum output fidelity* between Ψ_0 and Ψ_1 is defined as

$$\mathrm{F}_{\max}(\Psi_0,\Psi_1) = \max\left\{\mathrm{F}(\Psi_0(\rho_0),\Psi_1(\rho_1)) \,:\, \rho_0,\rho_1 \in \mathrm{D}(\mathcal{X})\right\}.$$

The characterization (which appears as an exercise in [8] and is a corollary of a slightly more general result proved in [13]) is given by the following theorem. A short proof is included for the sake of completeness.

Theorem 2.4. *Let $\mathcal{X} = \mathbb{C}^n$, $\mathcal{Y} = \mathbb{C}^m$, and $\mathcal{Z} = \mathbb{C}^k$, let $A_0, A_1 \in \mathrm{L}(\mathcal{X}, \mathcal{Y}\otimes\mathcal{Z})$ be operators, and let $\Psi_0,\Psi_1 \in \mathrm{T}(\mathcal{X},\mathcal{Z})$ and $\Phi \in \mathrm{T}(\mathcal{X},\mathcal{Y})$ be mappings defined by the equations*

$$\Psi_0(X) = \mathrm{Tr}_{\mathcal{Y}}\left(A_0XA_0^*\right), \qquad \Psi_1(X) = \mathrm{Tr}_{\mathcal{Y}}\left(A_1XA_1^*\right), \qquad \textit{and} \qquad \Phi(X) = \mathrm{Tr}_{\mathcal{Z}}\left(A_0XA_1^*\right),$$

for all $X \in \mathrm{L}(\mathcal{X})$. It holds that $|||\Phi|||_1 = \mathrm{F}_{\max}(\Psi_0,\Psi_1)$.

Proof. For $\mathcal{W} = \mathbb{C}^n$ and any choice of vectors $u_0, u_1 \in \mathcal{X} \otimes \mathcal{W}$, one has

$$\operatorname{Tr}_{\mathcal{Y}\otimes\mathcal{W}}\big((A_0 \otimes \mathbb{1}_{\mathcal{W}})u_0u_0^*(A_0 \otimes \mathbb{1}_{\mathcal{W}})^*\big) = \Psi_0\big(\operatorname{Tr}_{\mathcal{W}}\big(u_0u_0^*\big)\big),$$
$$\operatorname{Tr}_{\mathcal{Y}\otimes\mathcal{W}}\big((A_1 \otimes \mathbb{1}_{\mathcal{W}})u_1u_1^*(A_1 \otimes \mathbb{1}_{\mathcal{W}})^*\big) = \Psi_1\big(\operatorname{Tr}_{\mathcal{W}}\big(u_1u_1^*\big)\big),$$

and therefore, by (1.4), it holds that

$$\big\|\operatorname{Tr}_{\mathcal{Z}}\big((A_0 \otimes \mathbb{1}_{\mathcal{W}})u_0u_1^*(A_1 \otimes \mathbb{1}_{\mathcal{W}})^*\big)\big\|_1 = \mathrm{F}\big(\Psi_0(\operatorname{Tr}_{\mathcal{W}}(u_0u_0^*)), \Psi_1(\operatorname{Tr}_{\mathcal{W}}(u_1u_1^*))\big).$$

Consequently

$$\begin{aligned}
|||\Phi|||_1 &= \max\big\{\big\|\operatorname{Tr}_{\mathcal{Z}}\big((A_0 \otimes \mathbb{1}_{\mathcal{W}})u_0u_1^*(A_1 \otimes \mathbb{1}_{\mathcal{W}})^*\big)\big\|_1 \,:\, u_0, u_1 \in \mathcal{X} \otimes \mathcal{W},\, \|u_0\| = \|u_1\| = 1\big\}\\
&= \max\big\{\mathrm{F}\big(\Psi_0(\operatorname{Tr}_{\mathcal{W}}(u_0u_0^*)), \Psi_1(\operatorname{Tr}_{\mathcal{W}}(u_1u_1^*))\big) : u_0, u_1 \in \mathcal{X} \otimes \mathcal{W},\, \|u_0\| = \|u_1\| = 1\big\}\\
&= \max\big\{\mathrm{F}(\Psi_0(\rho_0), \Psi_1(\rho_1)) \,:\, \rho_0, \rho_1 \in \mathrm{D}(\mathcal{X})\big\}\\
&= \mathrm{F}_{\max}(\Psi_0, \Psi_1)
\end{aligned}$$

as required. □

2.3 A semidefinite program for the maximum output fidelity

Theorem 2.4, when combined with the semidefinite program for the fidelity discussed at the beginning of the present section, leads to a semidefinite program for the completely bounded trace norm, as is now described.

Let $\mathcal{X} = \mathbb{C}^n$ and $\mathcal{Y} = \mathbb{C}^m$, and suppose that a mapping $\Phi \in \mathrm{T}(\mathcal{X}, \mathcal{Y})$ is given as

$$\Phi(X) = \operatorname{Tr}_{\mathcal{Z}}\big(A_0XA_1^*\big) \tag{2.2}$$

for all $X \in \mathrm{L}(\mathcal{X})$, where $\mathcal{Z} = \mathbb{C}^k$ and $A_0, A_1 \in \mathrm{L}(\mathcal{X}, \mathcal{Y} \otimes \mathcal{Z})$ are operators. An expression of this form is sometimes known as a *Stinespring representation* (or a *generalized Stinespring representation*) of Φ. It is a simple exercise to show that such a representation of a mapping Φ always exists, provided that k is sufficiently large (at least mn in the worst case). As in Theorem 2.4, define mappings $\Psi_0, \Psi_1 \in \mathrm{T}(\mathcal{X}, \mathcal{Z})$ as

$$\Psi_0(X) = \operatorname{Tr}_{\mathcal{Y}}\big(A_0XA_0^*\big) \qquad \text{and} \qquad \Psi_1(X) = \operatorname{Tr}_{\mathcal{Y}}\big(A_1XA_1^*\big)$$

for all $X \in \mathrm{L}(\mathcal{X})$.

The primal and dual problems associated with the semidefinite program for the maximum output fidelity of Ψ_0 and Ψ_1 (which agrees with the value $|||\Phi|||_1$ by Theorem 2.4) may be expressed as follows:

Primal problem

$$\begin{aligned}
\text{maximize:}\quad & \frac{1}{2}\operatorname{Tr}(X) + \frac{1}{2}\operatorname{Tr}(X^*)\\
\text{subject to:}\quad & \begin{pmatrix} \Psi_0(\rho_0) & X \\ X^* & \Psi_1(\rho_1) \end{pmatrix} \geq 0\\
& \rho_0, \rho_1 \in \mathrm{D}(\mathcal{X})\\
& X \in \mathrm{L}(\mathcal{Z}).
\end{aligned}$$

Dual problem

$$\begin{aligned}
\text{minimize:}\quad & \frac{1}{2}\big\|\Psi_0^*(Y)\big\|_\infty + \frac{1}{2}\big\|\Psi_1^*(Y^{-1})\big\|_\infty\\
\text{subject to:}\quad & Y \in \mathrm{Pd}(\mathcal{Z}).
\end{aligned}$$

A more formal specification of the semidefinite program is given by the triple (Ξ, C, D), where $\Xi : \mathrm{L}(\mathcal{X}\oplus\mathcal{X}\oplus\mathcal{Z}\oplus\mathcal{Z}) \to \mathrm{L}(\mathbb{C}\oplus\mathbb{C}\oplus\mathcal{Z}\oplus\mathcal{Z})$ is a Hermiticity-preserving mapping defined as

$$\Xi\begin{pmatrix} W_0 & \cdot & \cdot & \cdot \\ \cdot & W_1 & \cdot & \cdot \\ \cdot & \cdot & Z_0 & \cdot \\ \cdot & \cdot & \cdot & Z_1 \end{pmatrix} = \begin{pmatrix} \operatorname{Tr}(W_0) & 0 & 0 & 0 \\ 0 & \operatorname{Tr}(W_1) & 0 & 0 \\ 0 & 0 & Z_0 - \Psi_0(W_0) & 0 \\ 0 & 0 & 0 & Z_1 - \Psi_1(W_1) \end{pmatrix} \tag{2.3}$$

for all $W_0, W_1 \in \mathrm{L}(\mathcal{X})$ and $Z_0, Z_1 \in \mathrm{L}(\mathcal{Z})$ (and again dots represent operators, on appropriately chosen spaces, upon which Ξ does not depend), and $C \in \operatorname{Herm}(\mathcal{X}\oplus\mathcal{X}\oplus\mathcal{Z}\oplus\mathcal{Z})$ and $D \in \operatorname{Herm}(\mathbb{C}\oplus\mathbb{C}\oplus\mathcal{Z}\oplus\mathcal{Z})$ are defined as

$$C = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbb{1} \\ 0 & 0 & \mathbb{1} & 0 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.4}$$

The adjoint of the mapping Ξ is given by

$$\Xi^*\begin{pmatrix} \lambda_0 & \cdot & \cdot & \cdot \\ \cdot & \lambda_1 & \cdot & \cdot \\ \cdot & \cdot & Y_0 & \cdot \\ \cdot & \cdot & \cdot & Y_1 \end{pmatrix} = \begin{pmatrix} \lambda_0\mathbb{1} - \Psi_0^*(Y_0) & 0 & 0 & 0 \\ 0 & \lambda_1\mathbb{1} - \Psi_1^*(Y_1) & 0 & 0 \\ 0 & 0 & Y_0 & 0 \\ 0 & 0 & 0 & Y_1 \end{pmatrix}.$$

After a simplification of the primal and dual problems associated with (Ξ, C, D), one obtains the problems given previously.

Strong duality

To prove that strong duality holds for the semidefinite program above, it suffices to prove that the primal problem is feasible and the dual problem is strictly feasible. Primal feasibility is easily checked: one may verify that the operator

$$\begin{pmatrix} \rho_0 & 0 & 0 & 0 \\ 0 & \rho_1 & 0 & 0 \\ 0 & 0 & \Psi_0(\rho_0) & 0 \\ 0 & 0 & 0 & \Psi_1(\rho_1) \end{pmatrix}$$

is primal feasible for any choice of density operators $\rho_0, \rho_1 \in \mathrm{D}(\mathcal{X})$. To verify that strict dual feasibility holds, one may consider the operator

$$\begin{pmatrix} \lambda_0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \mathbb{1} & 0 \\ 0 & 0 & 0 & \mathbb{1} \end{pmatrix}$$

for any choice of real numbers $\lambda_0 > \|\Psi_0^*(\mathbb{1})\|_\infty$ and $\lambda_1 > \|\Psi_1^*(\mathbb{1})\|_\infty$. By Slater's theorem, strong duality follows.

Optimal value

For any fixed choice of $\rho_0, \rho_1 \in \mathrm{D}(\mathcal{X})$, one has that the maximum value of the primal objective function

$$\frac{1}{2}\operatorname{Tr}(X) + \frac{1}{2}\operatorname{Tr}(X^*)$$

subject to the constraint

$$\begin{pmatrix} \Psi_0(\rho_0) & X \\ X^* & \Psi_1(\rho_1) \end{pmatrix} \geq 0$$

is equal to $\mathrm{F}(\Psi_0(\rho_0), \Psi_1(\rho_1))$, by the same analysis that was used to determine the primal optimum for the semidefinite program for the fidelity function. Maximizing over all choices of density operators $\rho_0, \rho_1 \in \mathrm{D}(\mathcal{X})$ gives $\mathrm{F}_{\max}(\Psi_0, \Psi_1)$, which equals $|||\Phi|||_1$ by Theorem 2.4.

3 A semidefinite program for the completely bounded trace norm from a mapping's Choi-Jamiołkowski representation

In this section an alternate semidefinite program for the completely bounded trace norm is presented. Whereas the semidefinite program from the previous section is obtained from a Stinespring representation of a given mapping, the semidefinite program in this section is obtained from the Choi-Jamiołkowski representation of a given mapping. (It should be noted that it is possible to efficiently convert between the Choi-Jamiołkowski and Stinespring representations of mappings, as well as other representations such as the Kraus representation, so either semidefinite program could be applied to a given mapping.)

While the two semidefinite programming formulations are different, they are related. As for the semidefinite programs for the fidelity and the completely bounded trace norm in the previous section, Lemma 2.1 provides a key tool through which the semidefinite program given in this section may be analyzed.

3.1 Choi-Jamiołkowski representations and the completely bounded trace norm

Let $\mathcal{X} = \mathbb{C}^n$ and $\mathcal{Y} = \mathbb{C}^m$, and assume that $\Phi \in \mathrm{T}(\mathcal{X}, \mathcal{Y})$ is a given mapping. The *Choi-Jamiołkowski representation* of Φ is the operator $J(\Phi) \in \mathrm{L}(\mathcal{Y} \otimes \mathcal{X})$ defined as

$$J(\Phi) = \sum_{1 \leq i,j \leq n} \Phi(E_{i,j}) \otimes E_{i,j}.$$

An equivalent expression is

$$J(\Phi) = (\Phi \otimes \mathbb{1}_{\mathrm{L}(\mathcal{X})})(\operatorname{vec}(\mathbb{1}_{\mathcal{X}}) \operatorname{vec}(\mathbb{1}_{\mathcal{X}})^*),$$

where the vec-mapping is the linear mapping defined by the action

$$\operatorname{vec}(E_{i,j}) = e_i \otimes e_j,$$

extended by linearity to arbitrary operators.

One identity connecting the vec-mapping to the Choi-Jamiołkowski representation of a mapping is the following one, which holds for all choices of $A, B \in \mathrm{L}(\mathcal{X})$:

$$\big(\mathbb{1}_{\mathcal{Y}} \otimes A^{\mathsf{T}}\big) J(\Phi) \big(\mathbb{1}_{\mathcal{Y}} \otimes \overline{B}\big) = \big(\Phi \otimes \mathbb{1}_{\mathrm{L}(\mathcal{X})}\big)\big(\operatorname{vec}(A)\operatorname{vec}(B)^*\big). \tag{3.1}$$

(Here, the notation A^{T} refers to the transpose of A, while $\overline{B}$ refers to the entry-wise complex conjugate of B.) Through this identity, an alternate expression for the completely bounded trace norm is obtained, as stated by the following theorem.

Theorem 3.1. *Let $\mathcal{X} = \mathbb{C}^n$ and $\mathcal{Y} = \mathbb{C}^m$, and let $\Phi \in \mathrm{T}(\mathcal{X}, \mathcal{Y})$ be a linear mapping. It holds that*

$$|||\Phi|||_1 = \max\Big\{ \big\| \big(\mathbb{1}_{\mathcal{Y}} \otimes \sqrt{\rho_0}\big) J(\Phi) \big(\mathbb{1}_{\mathcal{Y}} \otimes \sqrt{\rho_1}\big) \big\|_1 \,:\, \rho_0, \rho_1 \in \mathrm{D}(\mathcal{X}) \Big\}.$$

Proof. By (1.2) together with (3.1) it holds that

$$|||\Phi|||_1 = \max\big\{ \big\| \big(\mathbb{1}_{\mathcal{Y}} \otimes A^{\mathsf{T}}\big) J(\Phi) \big(\mathbb{1}_{\mathcal{Y}} \otimes \overline{B}\big) \big\|_1 \,:\, A, B \in \mathrm{L}(\mathcal{X}),\, \|A\|_2 = \|B\|_2 = 1 \big\}.$$

By the polar decomposition, every operator $X \in \mathrm{L}(\mathcal{X})$ with $\|X\|_2 = 1$ may be written as $X = \sqrt{\sigma} U$ for some choice of $\sigma \in \mathrm{D}(\mathcal{X})$ and $U \in \mathrm{U}(\mathcal{X})$. By the unitary invariance of the trace norm, the theorem follows. □

3.2 A semidefinite program from Theorem 3.1

The following primal and dual problems are associated with the semidefinite program for $|||\Phi|||_1$ that we derive from Theorem 3.1:

Primal problem

$$\begin{aligned}
\text{maximize:}\quad & \frac{1}{2}\langle J(\Phi), X\rangle + \frac{1}{2}\langle J(\Phi)^*, X^*\rangle \\
\text{subject to:}\quad & \begin{pmatrix} \mathbb{1}_{\mathcal{Y}} \otimes \rho_0 & X \\ X^* & \mathbb{1}_{\mathcal{Y}} \otimes \rho_1 \end{pmatrix} \geq 0 \\
& \rho_0, \rho_1 \in \mathrm{D}(\mathcal{X}) \\
& X \in \mathrm{L}(\mathcal{Y} \otimes \mathcal{X})
\end{aligned}$$

Dual problem

$$\begin{aligned}
\text{minimize:}\quad & \frac{1}{2}\big\|\operatorname{Tr}_{\mathcal{Y}}(Y_0)\big\|_\infty + \frac{1}{2}\big\|\operatorname{Tr}_{\mathcal{Y}}(Y_1)\big\|_\infty \\
\text{subject to:}\quad & \begin{pmatrix} Y_0 & -J(\Phi) \\ -J(\Phi)^* & Y_1 \end{pmatrix} \geq 0 \\
& Y_0, Y_1 \in \mathrm{Pos}(\mathcal{Y} \otimes \mathcal{X})
\end{aligned}$$

The semidefinite program giving rise to these problems may be specified more formally by the triple (Ξ, C, D), where

$$\Xi \in \mathrm{T}(\mathcal{X} \oplus \mathcal{X} \oplus (\mathcal{Y} \otimes \mathcal{X}) \oplus (\mathcal{Y} \otimes \mathcal{X}), \mathbb{C} \oplus \mathbb{C} \oplus (\mathcal{Y} \otimes \mathcal{X}) \oplus (\mathcal{Y} \otimes \mathcal{X}))$$

is a Hermiticity-preserving mapping defined as

$$\Xi \begin{pmatrix} W_0 & \cdot & \cdot & \cdot \\ \cdot & W_1 & \cdot & \cdot \\ \cdot & \cdot & Z_0 & \cdot \\ \cdot & \cdot & \cdot & Z_1 \end{pmatrix} = \begin{pmatrix} \operatorname{Tr}(W_0) & 0 & 0 & 0 \\ 0 & \operatorname{Tr}(W_1) & 0 & 0 \\ 0 & 0 & Z_0 - \mathbb{1} \otimes W_0 & 0 \\ 0 & 0 & 0 & Z_1 - \mathbb{1} \otimes W_1 \end{pmatrix} \tag{3.2}$$

and $C \in \text{Herm}(\mathcal{X} \oplus \mathcal{X} \oplus (\mathcal{Y} \otimes \mathcal{X}) \oplus (\mathcal{Y} \otimes \mathcal{X}))$ and $D \in \text{Herm}(\mathbb{C} \oplus \mathbb{C} \oplus (\mathcal{Y} \otimes \mathcal{X}) \oplus (\mathcal{Y} \otimes \mathcal{X}))$ are defined as

$$C = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & J(\Phi) \\ 0 & 0 & J(\Phi)^* & 0 \end{pmatrix} \qquad \text{and} \qquad D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.3}$$

The adjoint of the mapping Ξ is given by

$$\Xi^*\begin{pmatrix} \lambda_0 & \cdot & \cdot & \cdot \\ \cdot & \lambda_1 & \cdot & \cdot \\ \cdot & \cdot & Y_0 & \cdot \\ \cdot & \cdot & \cdot & Y_1 \end{pmatrix} = \begin{pmatrix} \lambda_0 \mathbb{1} - \text{Tr}_{\mathcal{Y}}(Y_0) & 0 & 0 & 0 \\ 0 & \lambda_1 \mathbb{1} - \text{Tr}_{\mathcal{Y}}(Y_1) & 0 & 0 \\ 0 & 0 & Y_0 & 0 \\ 0 & 0 & 0 & Y_1 \end{pmatrix}.$$

After a simplification of the primal and dual problems associated with (Ξ, C, D), one obtains the problems given above.

Strong duality

Similar to the semidefinite programs discussed in the previous section, strong duality is easily established for the semidefinite program described above by the use of Slater's theorem. In fact, strict primal and strict dual feasibility hold for all choices of Φ; in addition to strong duality, the primal and dual optima are achieved by feasible solutions in both cases. An example of a strictly feasible primal solution is

$$\frac{1}{\dim(\mathcal{X})}\begin{pmatrix} \mathbb{1}_{\mathcal{X}} & 0 & 0 & 0 \\ 0 & \mathbb{1}_{\mathcal{X}} & 0 & 0 \\ 0 & 0 & \mathbb{1}_{\mathcal{Y}} \otimes \mathbb{1}_{\mathcal{X}} & 0 \\ 0 & 0 & 0 & \mathbb{1}_{\mathcal{Y}} \otimes \mathbb{1}_{\mathcal{X}} \end{pmatrix},$$

while an example of a strictly feasible dual solution is

$$\begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & Y & 0 \\ 0 & 0 & 0 & Y \end{pmatrix}$$

for any choice of Y and λ with

$$Y > \frac{1}{2}\|J(\Phi)\|_\infty \mathbb{1}_{\mathcal{Y}} \otimes \mathbb{1}_{\mathcal{X}} \qquad \text{and} \qquad \lambda > \left\|\text{Tr}_{\mathcal{Y}}(Y)\right\|_\infty.$$

Optimal value

For any choice of density operators $\rho_0, \rho_1 \in \text{D}(\mathcal{X})$, it holds that

$$\begin{pmatrix} \mathbb{1}_{\mathcal{Y}} \otimes \rho_0 & X \\ X^* & \mathbb{1}_{\mathcal{Y}} \otimes \rho_1 \end{pmatrix} \geq 0 \tag{3.4}$$

if and only if

$$X = \big(\mathbb{1}_{\mathcal{Y}} \otimes \sqrt{\rho_0}\big) K \big(\mathbb{1}_{\mathcal{Y}} \otimes \sqrt{\rho_1}\big) \tag{3.5}$$

for some choice of an operator $K \in \mathrm{L}(\mathcal{Y} \otimes \mathcal{X})$ satisfying $\|K\|_\infty \leq 1$, as follows from Lemma 2.1. The primal optimum is therefore given by

$$\begin{aligned}\sup_{K,\rho_0,\rho_1} \Re\Big(\big\langle J(\Phi), \big(\mathbb{1}_{\mathcal{Y}} \otimes \sqrt{\rho_0}\big) K \big(\mathbb{1}_{\mathcal{Y}} \otimes \sqrt{\rho_1}\big)\big\rangle\Big) &= \sup_{\rho_0,\rho_1} \Big\| \big(\mathbb{1}_{\mathcal{Y}} \otimes \sqrt{\rho_1}\big) J(\Phi)^* \big(\mathbb{1}_{\mathcal{Y}} \otimes \sqrt{\rho_0}\big) \Big\|_1 \\ &= \sup_{\rho_0,\rho_1} \Big\| \big(\mathbb{1}_{\mathcal{Y}} \otimes \sqrt{\rho_0}\big) J(\Phi) \big(\mathbb{1}_{\mathcal{Y}} \otimes \sqrt{\rho_1}\big) \Big\|_1 = |||\Phi|||_1,\end{aligned}$$

where supremums are taken over all $K \in \mathrm{L}(\mathcal{X})$ with $\|K\|_\infty \leq 1$ and $\rho_0, \rho_1 \in \mathrm{D}(\mathcal{X})$, and where the last equality follows from Theorem 3.1.

4 Remarks on the complexity of approximating optimal solutions to the semidefinite programs

Suppose that (Ξ, C, D) is an instance of one of the semidefinite programs described above, either for the maximum output fidelity characterization or the Choi-Jamiołkowski representation characterization of the completely bounded trace norm. It is natural to ask whether an approximation to the optimal value of this semidefinite program can be efficiently computed (under the assumption, let us say, that the complex numbers specifying Ξ, C, and D have rational real and imaginary parts whose numerators and denominators are represented as integers in binary notation).

From a practical viewpoint, algorithms employing *interior point methods* represent a sensible approach for computing the optimum value of these semidefinite programs [1, 4]. The CVX software package [5] for the MATLAB numerical computing environment allows one to solve these semidefinite programs efficiently with minimal coding requirements.

For the sake of obtaining rigorous statements about the polynomial-time solvability of the semidefinite programs (and perhaps not much more than that), the *ellipsoid method* is a more attractive alternative. In principle, this method can be applied to the primal or dual formulation of a semidefinite program, but for the semidefinite programs in this paper it is easier to work with the dual problems. As is described in detail in [6] for a significantly more general setting, and summarized in [9] for the semidefinite programming setting, the bounds derived below allow one to conclude that an algorithm running in time polynomial in the input size and $\log(1/\delta)$ can approximate the optimal value of the semidefinite programs discussed above to within accuracy δ (for any choice of $\delta > 0$).

A ball in the interior of the feasible region

For $\mathcal{D} \subseteq \mathrm{Herm}(\mathcal{Y})$ denoting the dual feasible set of a given semidefinite program (Ξ, C, D) and $\varepsilon > 0$ being a positive real number, one defines

$$\mathcal{D}_\varepsilon^\circ = \big\{ Y \in \mathrm{Herm}(\mathcal{Y}) : Y + H \in \mathcal{D} \text{ for all } H \in \mathrm{Herm}(\mathcal{Y}) \text{ satisfying } \|H\|_2 \leq \varepsilon \big\}.$$

Intuitively speaking, $\mathcal{D}_\varepsilon^\circ$ contains every operator in the interior of the dual feasible set that is not too close to the boundary of that set.

It has already been demonstrated that $\mathcal{D}_\varepsilon^\circ$ is nonempty for some choice of ε for each of the semidefinite programs, in the discussions of strong duality in the two previous sections. To argue that accurate approximate solutions to the semidefinite programs can be obtained by the ellipsoid method as stated above, the first thing that is required is a sufficiently large lower bound on the value of ε for which $\mathcal{D}_\varepsilon^\circ$ is nonempty.

For the semidefinite program for the maximum output fidelity characterization of the completely bounded trace norm, presented in Section 2, the adjoint of the mapping Ξ is given by

$$\Xi^*\begin{pmatrix}\lambda_0 & \cdot & \cdot & \cdot\\ \cdot & \lambda_1 & \cdot & \cdot\\ \cdot & \cdot & Y_0 & \cdot\\ \cdot & \cdot & \cdot & Y_1\end{pmatrix} = \begin{pmatrix}\lambda_0\mathbb{1}_{\mathcal{X}} - \Psi_0^*(Y_0) & 0 & 0 & 0\\ 0 & \lambda_1\mathbb{1}_{\mathcal{X}} - \Psi_1^*(Y_1) & 0 & 0\\ 0 & 0 & Y_0 & 0\\ 0 & 0 & 0 & Y_1\end{pmatrix}.$$

The operator

$$\begin{pmatrix}\lambda_0 & 0 & 0 & 0\\ 0 & \lambda_1 & 0 & 0\\ 0 & 0 & \mathbb{1} & 0\\ 0 & 0 & 0 & \mathbb{1}\end{pmatrix}$$

for

$$\lambda_0 = \frac{1}{2} + \|\Psi_0^*(\mathbb{1})\|_\infty \qquad \text{and} \qquad \lambda_1 = \frac{1}{2} + \|\Psi_1^*(\mathbb{1})\|_\infty$$

is a specific example of a strictly dual feasible solution satisfying

$$\Xi^*\begin{pmatrix}\lambda_0 & \cdot & \cdot & \cdot\\ \cdot & \lambda_1 & \cdot & \cdot\\ \cdot & \cdot & \mathbb{1} & \cdot\\ \cdot & \cdot & \cdot & \mathbb{1}\end{pmatrix} - \frac{1}{2}\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & \mathbb{1}\\ 0 & 0 & \mathbb{1} & 0\end{pmatrix} \geq \frac{1}{2}\begin{pmatrix}\mathbb{1} & 0 & 0 & 0\\ 0 & \mathbb{1} & 0 & 0\\ 0 & 0 & \mathbb{1} & 0\\ 0 & 0 & 0 & \mathbb{1}\end{pmatrix}.$$

A calculation reveals that for $H \in \operatorname{Herm}(\mathbb{C} \oplus \mathbb{C} \oplus \mathcal{Z} \oplus \mathcal{Z})$ satisfying

$$\|H\|_2 \leq \frac{1}{4}\min\Big\{\|\Psi_0^*\|_\infty^{-1},\, \|\Psi_1^*\|_\infty^{-1},\, 1\Big\},$$

it holds that $\|\Xi^*(H)\|_\infty \leq 1/2$. As Ψ_0^* and Ψ_1^* are positive, it holds that $\|\Psi_0^*\|_\infty = \|\Psi_0^*(\mathbb{1})\|_\infty$ and $\|\Psi_1^*\|_\infty = \|\Psi_1^*(\mathbb{1})\|_\infty$, from which it follows that $\mathcal{D}_\varepsilon^\circ$ is nonempty for

$$\varepsilon = \frac{1}{4\big(1 + \|\Psi_0^*(\mathbb{1})\|_\infty + \|\Psi_1^*(\mathbb{1})\|_\infty\big)}.$$

For the semidefinite program for the completely bounded trace norm presented in Section 3, based on the Choi-Jamiołkowski representation, the adjoint of the mapping Ξ is given by

$$\Xi^*\begin{pmatrix}\lambda_0 & \cdot & \cdot & \cdot\\ \cdot & \lambda_1 & \cdot & \cdot\\ \cdot & \cdot & Y_0 & \cdot\\ \cdot & \cdot & \cdot & Y_1\end{pmatrix} = \begin{pmatrix}\lambda_0\mathbb{1}_{\mathcal{X}} - \operatorname{Tr}_{\mathcal{Y}}(Y_0) & 0 & 0 & 0\\ 0 & \lambda_1\mathbb{1}_{\mathcal{X}} - \operatorname{Tr}_{\mathcal{Y}}(Y_1) & 0 & 0\\ 0 & 0 & Y_0 & 0\\ 0 & 0 & 0 & Y_1\end{pmatrix}.$$

The operator

$$\begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & Y & 0 \\ 0 & 0 & 0 & Y \end{pmatrix}$$

for

$$Y = \left(\frac{\|J(\Phi)\|_\infty}{2} + 1\right)\mathbb{1}_{\mathcal{Y}\otimes\mathcal{X}}, \qquad \text{and} \qquad \lambda = 1 + \left(\frac{\|J(\Phi)\|_\infty}{2} + 1\right)\dim(\mathcal{Y})$$

is an example of a strictly dual feasible solution satisfying

$$\Xi^* \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & Y & 0 \\ 0 & 0 & 0 & Y \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & J(\Phi) \\ 0 & 0 & J(\Phi)^* & 0 \end{pmatrix} \geq \begin{pmatrix} \mathbb{1} & 0 & 0 & 0 \\ 0 & \mathbb{1} & 0 & 0 \\ 0 & 0 & \mathbb{1} & 0 \\ 0 & 0 & 0 & \mathbb{1} \end{pmatrix}.$$

For $H \in \text{Herm}(\mathbb{C} \oplus \mathbb{C} \oplus (\mathcal{Y}\otimes\mathcal{X}) \oplus (\mathcal{Y}\otimes\mathcal{X}))$ satisfying

$$\|H\|_2 \leq \frac{1}{2\dim(\mathcal{Y})}$$

it holds that $\|\Xi^*(H)\|_\infty \leq 1$, from which it follows that $\mathcal{D}_\varepsilon^\circ$ is nonempty for

$$\varepsilon = \frac{1}{2\dim(\mathcal{Y})}.$$

In both cases, the lower bound on the value of ε for which $\mathcal{D}_\varepsilon^\circ$ is nonempty is sufficiently large for the ellipsoid method to function as claimed above.

A bound on the size of an optimal solution

The second bound that is needed to prove the efficiency of the ellipsoid method is an upper bound on the size of an optimal, or near optimal, dual feasible solution. (This condition can be relaxed, but for the semidefinite programs in this paper it is not necessary to do so.)

For the semidefinite program based on the maximum output fidelity characterization of the completely bounded trace norm, an assumption on the mappings Ψ_0 and Ψ_1 will be made, which is that $\Psi_0(\mathbb{1})$ and $\Psi_1(\mathbb{1})$ are positive definite. In the case that a mapping $\Phi \in \text{T}(\mathcal{X},\mathcal{Y})$ is expressed as

$$\Phi(X) = \text{Tr}_{\mathcal{W}}(B_0 X B_1^*)$$

for all $X \in \text{L}(\mathcal{X})$, for some choice of a space $\mathcal{W} = \mathbb{C}^k$ and operators $B_0, B_1 \in \text{L}(\mathcal{X}, \mathcal{Y}\otimes\mathcal{W})$, it is routine to compute operators $A_0, A_1 \in \text{L}(\mathcal{X}, \mathcal{Y}\otimes\mathcal{Z})$, for $\mathcal{Z} = \mathbb{C}^r$ and $r = \text{rank}(J(\Phi))$, such that

$$\Phi(X) = \text{Tr}_{\mathcal{Z}}(A_0 X A_1^*)$$

for all $X \in \text{L}(\mathcal{X})$, and such that the mappings $\Psi_0, \Psi_1 \in \text{T}(\mathcal{X},\mathcal{Z})$ defined as

$$\Psi_0(X) = \text{Tr}_{\mathcal{Y}}(A_0 X A_0^*) \qquad \text{and} \qquad \Psi_1(X) = \text{Tr}_{\mathcal{Y}}(A_1 X A_1^*)$$

for all $X \in \mathrm{L}(\mathcal{X})$ satisfy $\Psi_0(\mathbb{1}) > 0$ and $\Psi_1(\mathbb{1}) > 0$. This assumption therefore causes no loss of generality with respect to the polynomial-time solvability of approximating the completely bounded trace norm of a mapping.

Suppose that an operator

$$\begin{pmatrix} \lambda_0 & \cdot & \cdot & \cdot \\ \cdot & \lambda_1 & \cdot & \cdot \\ \cdot & \cdot & Y_0 & \cdot \\ \cdot & \cdot & \cdot & Y_1 \end{pmatrix} \tag{4.1}$$

is a dual optimal solution. (Given that the primal problem is strictly feasible under the assumption that $\Psi_0(\mathbb{1})$ and $\Psi_1(\mathbb{1})$ are positive definite, the dual optimum will be achieved.) It follows that

$$\lambda_0 + \lambda_1 = |||\Phi|||_1 \leq \frac{1}{2}\|\Psi_0^*(\mathbb{1})\|_\infty + \frac{1}{2}\|\Psi_1^*(\mathbb{1})\|_\infty,$$

as well as

$$\Psi_0^*(Y_0) \leq \lambda_0 \mathbb{1} \qquad \text{and} \qquad \Psi_1^*(Y_1) \leq \lambda_1 \mathbb{1}.$$

It follows that

$$\mathrm{Tr}(Y_0) \leq \frac{1}{\lambda_{\min}(\Psi_0(\mathbb{1}))}\langle \Psi_0(\mathbb{1}), Y_0\rangle = \frac{1}{\lambda_{\min}(\Psi_0(\mathbb{1}))}\mathrm{Tr}(\Psi_0^*(Y_0)) \leq \frac{\lambda_0 \dim(\mathcal{X})}{\lambda_{\min}(\Psi_0(\mathbb{1}))},$$

where $\lambda_{\min}(\Psi_0(\mathbb{1}))$ denotes the smallest eigenvalue of $\Psi_0(\mathbb{1})$. By similar reasoning,

$$\mathrm{Tr}(Y_1) \leq \frac{\lambda_1 \dim(\mathcal{X})}{\lambda_{\min}(\Psi_1(\mathbb{1}))}.$$

The operator (4.1) must therefore have trace bounded by

$$R = \left(\frac{1}{2}\|\Psi_0^*(\mathbb{1})\|_\infty + \frac{1}{2}\|\Psi_1^*(\mathbb{1})\|_\infty\right)\left(1 + \frac{\dim(\mathcal{X})}{\lambda_{\min}(\Psi_0(\mathbb{1}))} + \frac{\dim(\mathcal{X})}{\lambda_{\min}(\Psi_1(\mathbb{1}))}\right).$$

As the operator (4.1) must also be positive semidefinite, its Frobenius norm is also upper-bounded by R.

For the semidefinite program for the Choi-Jamiołkowski representation characterization of the completely bounded trace norm, one may again consider a dual-optimal solution of the form (4.1) (where this time $Y_0, Y_1 \in \mathrm{Pos}(\mathcal{Y} \otimes \mathcal{X})$). It holds that

$$\lambda_0 + \lambda_1 = |||\Phi|||_1 \leq \|J(\Phi)\|_1,$$

as well as

$$\mathrm{Tr}_{\mathcal{Y}}(Y_0) \leq \lambda_0 \mathbb{1} \qquad \text{and} \qquad \mathrm{Tr}_{\mathcal{Y}}(Y_1) \leq \lambda_1 \mathbb{1}$$

The trace of any dual-optimal solution is therefore upper-bounded by

$$R = (\dim(\mathcal{X}) + 1)\|J(\Phi)\|_1.$$

Again, as every dual-feasible operator must be positive semidefinite, the Frobenius norm of any dual-optimal solution is also upper-bounded by R.

In both cases, the upper bound R is sufficient to argue that the ellipsoid method functions as claimed.

Acknowledgments

Thanks to Gus Gutoski for helpful comments on this paper. This research was supported by Canada's NSERC and the Canadian Institute for Advanced Research (CIFAR).

References

[1] F. ALIZADEH: Interior point methods in semidefinite programming with applications to combinatorial optimization. *SIAM Journal on Optimization*, 5(1):13–51, 1995. 14

[2] A. BEN-AROYA AND A. TA-SHMA: On the complexity of approximating the diamond norm. *Quantum Information and Computation*, 10(1):77–86, 2010. 5

[3] R. BHATIA: *Matrix Analysis*. Springer, 1997. 7

[4] E. DE KLERK: *Aspects of Semidefinite Programming – Interior Point Algorithms and Selected Applications*. Volume 65 of *Applied Optimization*. Kluwer Academic Publishers, Dordrecht, 2002. 14

[5] M. GRANT AND S. BOYD: CVX: Matlab software for disciplined convex programming. Available from http://stanford.edu/~boyd/cvx, 2009. 14

[6] M. GRÖTSCHEL, L. LOVÁSZ, AND A. SCHRIJVER: *Geometric Algorithms and Combinatorial Optimization*. Springer–Verlag, second corrected edition, 1993. 14

[7] N. KILLORAN: *Entanglement quantification and quantum benchmarking of optical communication devices*. PhD thesis, University of Waterloo, 2012. 5

[8] A. KITAEV, A. SHEN, AND M. VYALYI: *Classical and Quantum Computation*. Volume 47 of *Graduate Studies in Mathematics*. American Mathematical Society, 2002. 1, 8

[9] L. LOVÁSZ: Semidefinite programs and combinatorial optimization. *Recent Advances in Algorithms and Combinatorics*, 2003. 14

[10] M. A. NIELSEN AND I. L. CHUANG: *Quantum Computation and Quantum Information*. Cambridge University Press, 2000. 1

[11] V. PAULSEN: *Completely Bounded Maps and Operator Algebras*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2002. 1

[12] B. ROSGEN AND J. WATROUS: On the hardness of distinguishing mixed-state quantum computations. In *Proceedings of the 20th Annual Conference on Computational Complexity*, pp. 344–354, 2005. 4

[13] J. WATROUS: Distinguishing quantum operations having few Kraus operators. *Quantum Information and Computation*, 8(9):819–833, 2008. 4, 8

[14] J. WATROUS: Semidefinite programs for completely bounded norms. *Theory of Computing*, 5(11), 2009. 1, 2, 4, 8

AUTHOR

John Watrous
Institute for Quantum Computing and School of Computer Science
University of Waterloo
john.watrous@uwaterloo.ca
http://www.cs.uwaterloo.ca/~watrous