Understanding Quantum Information and Computation

Lesson 15

Quantum Code Constructions

John Watrous

Classical linear codes

Let $\Sigma = \{0, 1\}$ denote the binary alphabet.

A *classical linear code* is a non-empty set of binary strings $\mathcal{C} \subseteq \Sigma^n$ with this property:

$$u, v \in \mathcal{C} \;\Rightarrow\; u \oplus v \in \mathcal{C}$$

Example: 3-bit repetition code

The 3-bit repetition code $\{000, 111\}$ is a classical linear code.

Example: $[7, 4, 3]$-Hamming code

The $[7, 4, 3]$-Hamming code is the classical linear code containing these strings:

0000000	1100001	1010010	0110011
0110100	1010101	1100110	0000111
1111000	0011001	0101010	1001011
1001100	0101101	0011110	1111111

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Two natural ways to describe a classical linear code:

1. *Generators:* a minimal list of strings $u_1, \ldots, u_m \in \Sigma^n$ such that

$$\mathcal{C} = \{\alpha_1 u_1 \oplus \cdots \oplus \alpha_m u_m \,:\, \alpha_1, \ldots, \alpha_m \in \{0, 1\}\}$$

2. *Parity checks:* a minimal list of strings $v_1, \ldots, v_r \in \Sigma^n$ such that

$$\mathcal{C} = \{u \in \Sigma^n \,:\, u \cdot v_1 = \cdots = u \cdot v_r = 0\}$$

(where $u \cdot v$ is the binary dot product of u and v).

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1. Generators: 0110100, 1010010, 1100001, 1111000
2. Parity checks: 1111000, 1100110, 1010101

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Note: parity checks are equivalent to *stabilizer generators* containing only Z and $\mathbb{1}$ Pauli matrices.

Example: 3-bit repetition code

The 3-bit repetition code $\{000, 111\}$ is a classical linear code.

1. Generator: 111
2. Parity checks: 110, 011

Equivalently, the strings in this code are standard basis states for the stabilizer code with stabilizer generators $Z\,Z\,\mathbb{1}$ and $\mathbb{1}\,Z\,Z$.

CSS codes

Stabilizer generators containing only Z and $\mathbb{1}$ Pauli matrices are equivalent to parity checks.

Example: 3-bit repetition code

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Parity checks: 110, 011
Stabilizer generators: $Z\,Z\,\mathbb{1}$, $\mathbb{1}\,Z\,Z$

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\end{array}$$

Parity checks: 1111000, 1100110, 1010101
Stabilizer generators: $Z\,Z\,Z\,Z\,\mathbb{1}\,\mathbb{1}\,\mathbb{1}$, $Z\,Z\,\mathbb{1}\,\mathbb{1}\,Z\,Z\,\mathbb{1}$, $Z\,\mathbb{1}\,Z\,\mathbb{1}\,Z\,\mathbb{1}\,Z$

CSS codes

Stabilizer generators containing only Z and $\mathbb{1}$ Pauli matrices are equivalent to parity checks. These are called *Z stabilizer generators.*

Stabilizer generators containing only X and $\mathbb{1}$ Pauli matrices are also equivalent to parity checks — for the plus/minus basis $\{|+\rangle, |-\rangle\}$.

Example: $[7, 4, 3]$-Hamming code

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\end{array}$$

Parity checks: 1111000, 1100110, 1010101

The stabilizer generators $X\,X\,X\,X\,\mathbb{1}\,\mathbb{1}\,\mathbb{1}$, $X\,X\,\mathbb{1}\,\mathbb{1}\,X\,X\,\mathbb{1}$, $X\,\mathbb{1}\,X\,\mathbb{1}\,X\,\mathbb{1}\,X$ define a stabilizer code that includes these states:

$$\begin{array}{cccc}
|{+}{+}{+}{+}{+}{+}{+}\rangle & |{-}{-}{+}{+}{+}{+}{-}\rangle & |{-}{+}{-}{+}{+}{-}{+}\rangle & |{+}{-}{-}{+}{+}{-}{-}\rangle \\
|{+}{-}{-}{+}{-}{+}{+}\rangle & |{-}{+}{-}{+}{-}{+}{-}\rangle & |{-}{-}{+}{+}{-}{-}{+}\rangle & |{+}{+}{+}{+}{-}{-}{-}\rangle \\
|{-}{-}{-}{-}{+}{+}{+}\rangle & |{+}{+}{-}{-}{+}{+}{-}\rangle & |{+}{-}{+}{-}{+}{-}{+}\rangle & |{-}{+}{+}{-}{+}{-}{-}\rangle \\
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\end{array}$$

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Definition: CSS codes

Stabilizer codes that can be expressed using only Z stabilizer generators and X stabilizer generators are called *CSS codes.*

Example: e-bit stabilizer code

$$\begin{matrix} Z\,Z \\ X\,X \end{matrix}$$

The code space is the one-dimensional space spanned by

$$|\phi^+\rangle = \frac{|0\rangle|0\rangle + |1\rangle|1\rangle}{\sqrt{2}} = \frac{|+\rangle|+\rangle + |-\rangle|-\rangle}{\sqrt{2}}$$

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Example: 7-qubit Steane code

$$\begin{array}{ccccccc}
Z & Z & Z & Z & \mathbb{1} & \mathbb{1} & \mathbb{1}\\
Z & Z & \mathbb{1} & \mathbb{1} & Z & Z & \mathbb{1}\\
Z & \mathbb{1} & Z & \mathbb{1} & Z & \mathbb{1} & Z\\
X & X & X & X & \mathbb{1} & \mathbb{1} & \mathbb{1}\\
X & X & \mathbb{1} & \mathbb{1} & X & X & \mathbb{1}\\
X & \mathbb{1} & X & \mathbb{1} & X & \mathbb{1} & X
\end{array}$$

Example: 9-qubit Shor code

$$\begin{array}{ccccccccc}
Z & Z & \mathbb{1} & \mathbb{1} & \mathbb{1} & \mathbb{1} & \mathbb{1} & \mathbb{1} & \mathbb{1}\\
\mathbb{1} & Z & Z & \mathbb{1} & \mathbb{1} & \mathbb{1} & \mathbb{1} & \mathbb{1} & \mathbb{1}\\
\mathbb{1} & \mathbb{1} & \mathbb{1} & Z & Z & \mathbb{1} & \mathbb{1} & \mathbb{1} & \mathbb{1}\\
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X & X & X & X & X & X & \mathbb{1} & \mathbb{1} & \mathbb{1}\\
\mathbb{1} & \mathbb{1} & \mathbb{1} & X & X & X & X & X & X
\end{array}$$

Error detection and correction

Consider a CSS code.

- The Z stabilizer generators detect X errors but are oblivious to Z errors (and corrections).
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Suppose the following:

- The Z stabilizer generators allow for the correction of up to j X errors.
- The X stabilizer generators allow for the correction of up to k Z errors.

7-qubit Steane code

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Z & Z & Z & Z & \mathbb{1} & \mathbb{1} & \mathbb{1} \\
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\end{array}$$

Then the CSS code allows for the correction of *any error* on up to $\min\{j, k\}$ qubits — we can simply detect and correct X errors and Z errors on this many qubits separately.

Code spaces of CSS codes

Consider a CSS code on n qubits.

Let $z_1, \ldots, z_s \in \Sigma^n$ be parity checks corresponding to the Z stabilizer generators.

$$\begin{aligned}\mathcal{C}_Z &= \{u \in \Sigma^n \,:\, u \cdot z_1 = \cdots = u \cdot z_s = 0\} \\ \mathcal{D}_Z &= \{\alpha_1 z_1 \oplus \cdots \oplus \alpha_s z_s \,:\, \alpha_1, \ldots, \alpha_s \in \{0,1\}\}\end{aligned}$$

Let $x_1, \ldots, x_t \in \Sigma^n$ be parity checks corresponding to the X stabilizer generators.

$$\begin{aligned}\mathcal{C}_X &= \{u \in \Sigma^n \,:\, u \cdot x_1 = \cdots = u \cdot x_t = 0\} \\ \mathcal{D}_X &= \{\alpha_1 x_1 \oplus \cdots \oplus \alpha_t x_t \,:\, \alpha_1, \ldots, \alpha_t \in \{0,1\}\}\end{aligned}$$

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The code space of the CSS code is spanned by vectors of either of these forms:

$$|u \oplus \mathcal{D}_X\rangle = \frac{1}{\sqrt{2^t}} \sum_{v \in \mathcal{D}_X} |u \oplus v\rangle \qquad \text{(for } u \in \mathcal{C}_Z\text{)}$$
$$H^{\otimes n}|u \oplus \mathcal{D}_Z\rangle = \frac{1}{\sqrt{2^s}} \sum_{v \in \mathcal{D}_Z} H^{\otimes n}|u \oplus v\rangle \qquad \text{(for } u \in \mathcal{C}_X\text{)}$$

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X & \mathbb{1} & X & \mathbb{1} & X & \mathbb{1} & X
\end{array}$$

We could encode $|0\rangle$ and $|1\rangle$ as follows:

$$\begin{aligned}
|0\rangle \mapsto\ & |0000000\rangle + |0011110\rangle + |0101101\rangle\\
& + |0110011\rangle + |1001011\rangle + |1010101\rangle\\
& + |1100110\rangle + |1111000\rangle
\end{aligned}$$

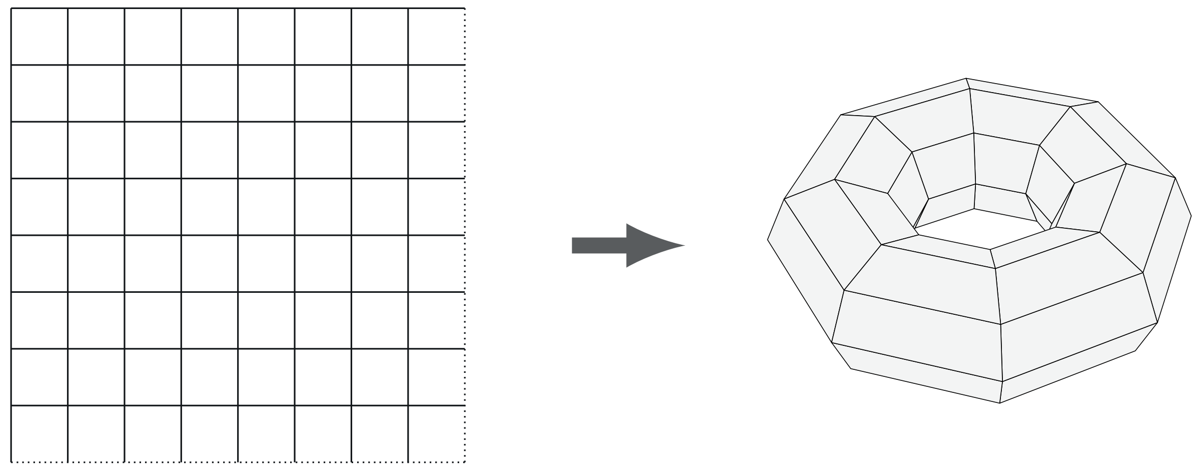
$$\begin{aligned}
|1\rangle \mapsto\ & |0000111\rangle + |0011001\rangle + |0101010\rangle\\
& + |0110100\rangle + |1001100\rangle + |1010010\rangle\\
& + |1100001\rangle + |1111111\rangle
\end{aligned}$$

Toric code

The *toric code* is an example of a quantum error correcting code (actually a family of codes) with a few key properties.

- Low weight stabilizer generators
- Geometric locality
- Large distance

Let $L \geq 2$ be a positive integer and consider an $L \times L$ lattice with periodic boundaries.



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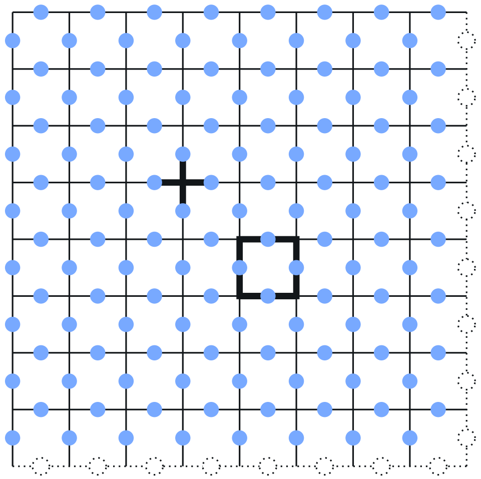
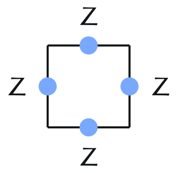


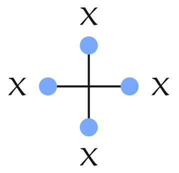
$L = 8$

Qubits are placed on the *edges* of the lattice
$\Rightarrow n = 2L^2$ qubits

There are two types of stabilizer generators:



Z stabilizer generators (tiles)



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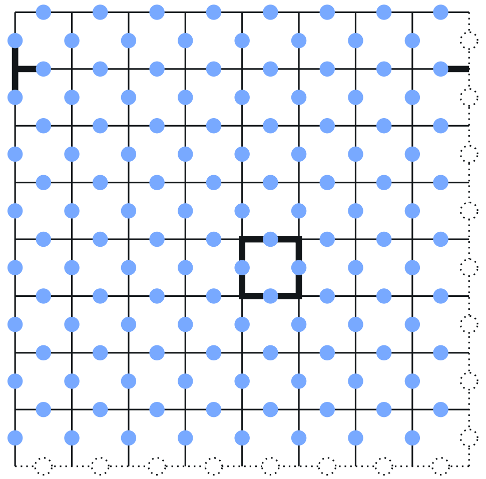

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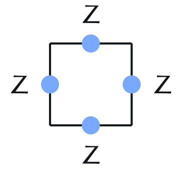

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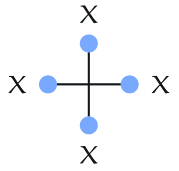

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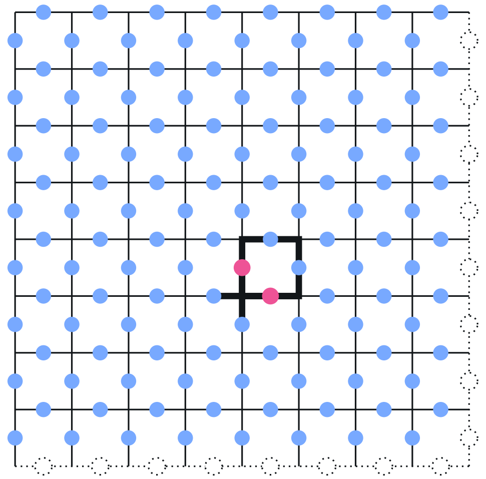

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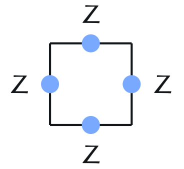
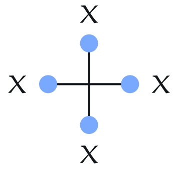
There are two types of stabilizer generators:

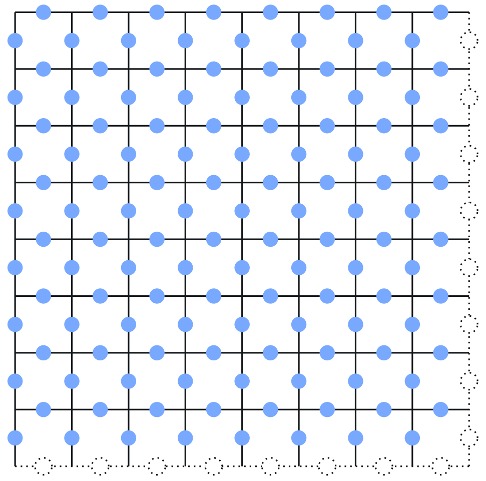


Z stabilizer generators (tiles)



X stabilizer generators (vertices)

Toric code



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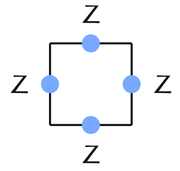

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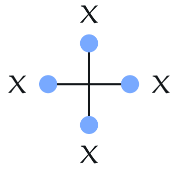


X stabilizer generators (vertices)

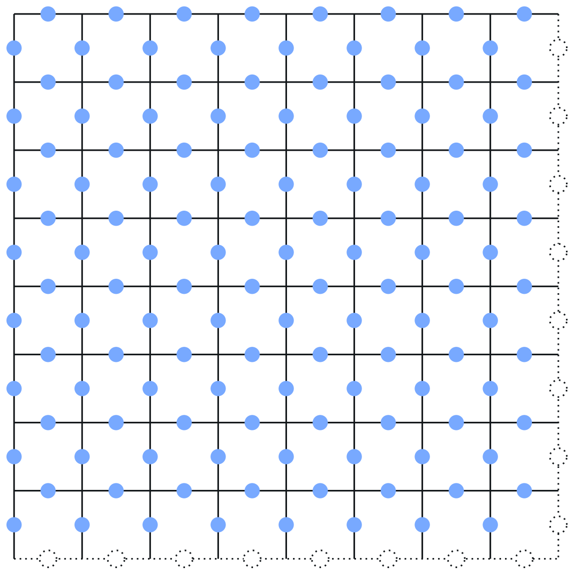
The product of all of the Z stabilizer generators is the identity — but removing any one leaves an independent set. Similar for the X stabilizer generators.

This leaves $L^2 - 1$ stabilizer generators of each of the two types.

The toric code (for this choice of L) therefore encodes $2L^2 - 2(L^2 - 1) = 2$ logical qubits into $2L^2$ physical qubits.

Errors and syndromes

The toric code is a CSS code, which allows us to consider X errors and Z errors separately. Let us focus on *X errors* — Z errors work similarly by symmetry.

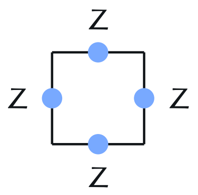


● unaffected qubit

● qubit affected by X error

□ +1 syndrome

■ −1 syndrome



Z stabilizer generator

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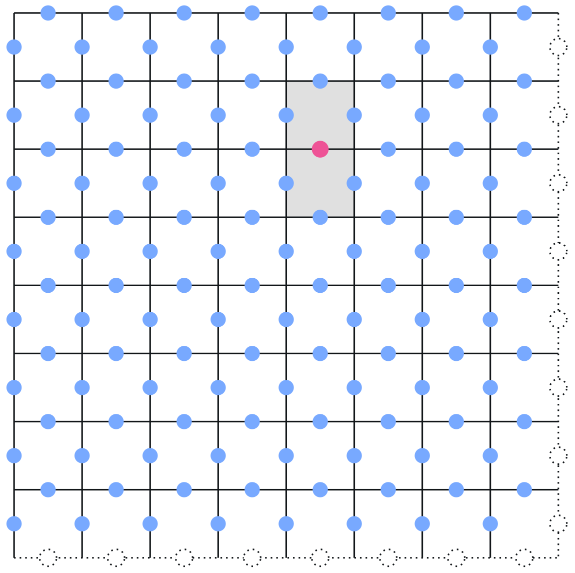

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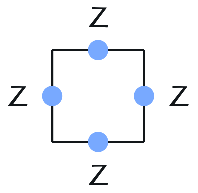

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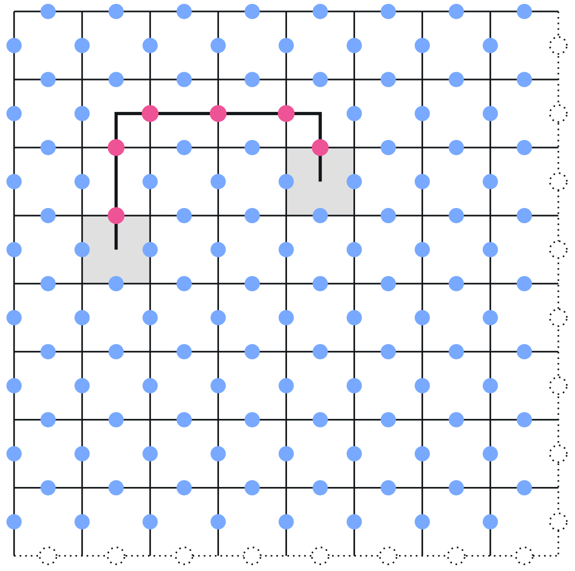

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Chains of adjacent X errors cause -1 syndrome outcomes at the *endpoints.*

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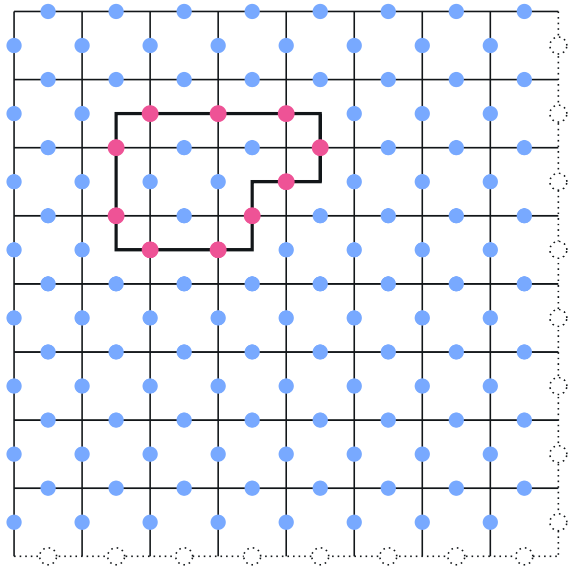

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Closed loops of adjacent X errors are undetected by the code.

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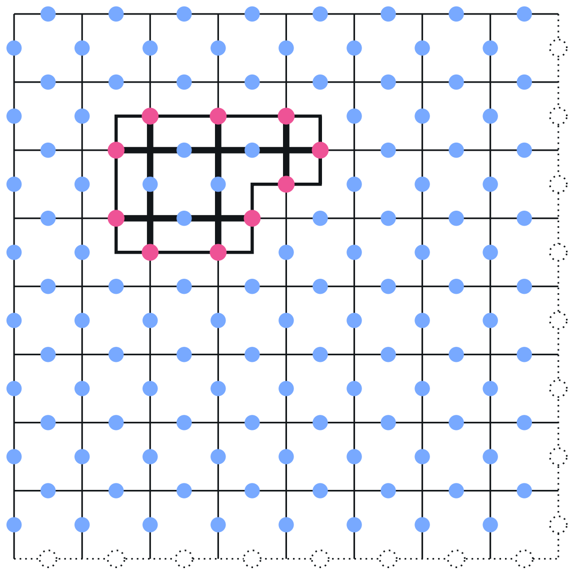

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- Loops crossing every line an even number of times are products of X stabilizer generators.

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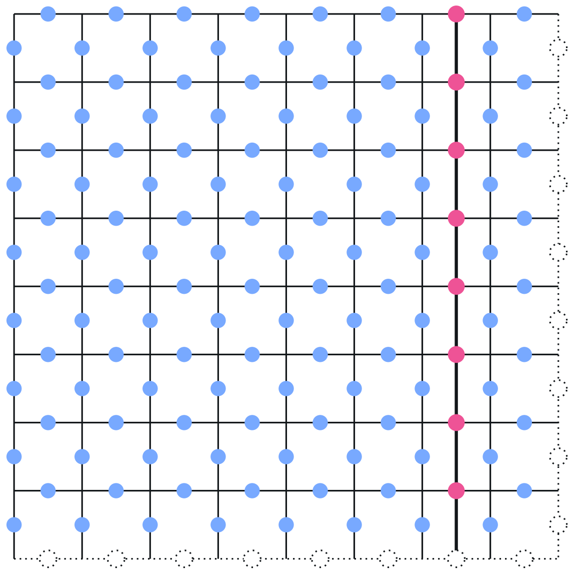

Chains of adjacent X errors cause -1 syndrome outcomes at the *endpoints.*

Closed loops of adjacent X errors are undetected by the code.

- Loops crossing every line an even number of times are products of X stabilizer generators.
- Loops crossing any line an odd number of times are nontrivial errors that are *undetected* by the code.

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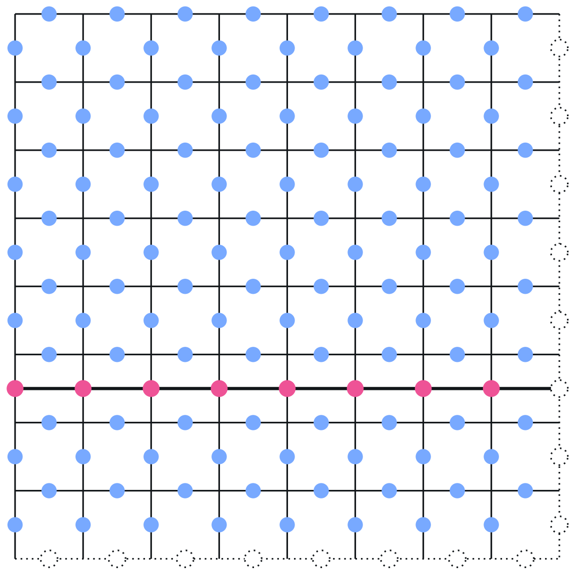


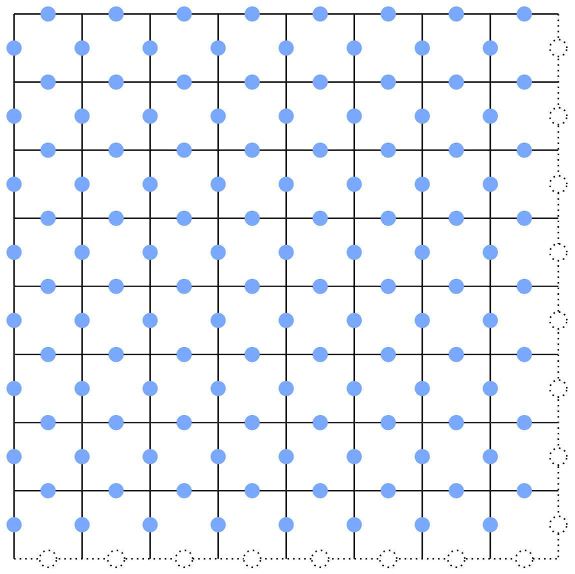
Chains of adjacent X errors cause -1 syndrome outcomes at the *endpoints.*

Closed loops of adjacent X errors are undetected by the code.

- Loops crossing every line an even number of times are products of X stabilizer generators.
- Loops crossing any line an odd number of times are nontrivial errors that are *undetected* by the code.

Errors and syndromes

The toric code is a CSS code, which allows us to consider X errors and Z errors separately. Let us focus on *X errors* — Z errors work similarly by symmetry.



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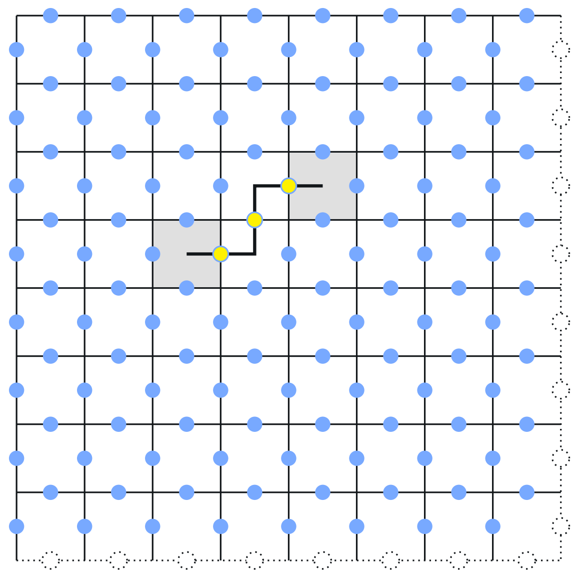
The minimum weight of a nontrivial, undetectable error is L.

The toric code is therefore a $[[2L^2, 2, L]]$ stabilizer code.

Correcting errors

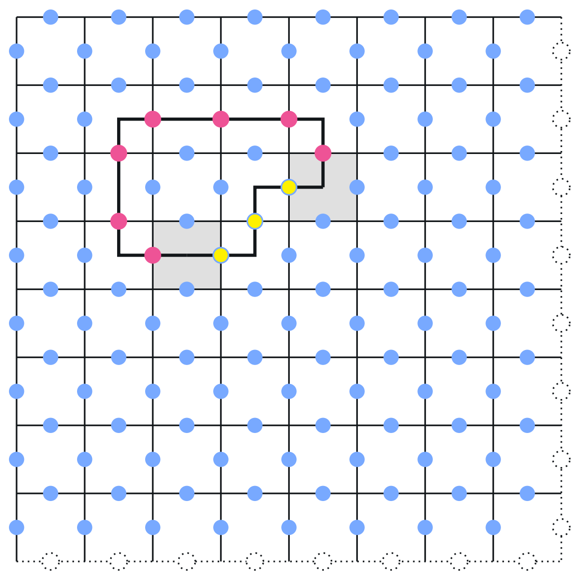


- ● unaffected qubit
- ● qubit affected by X error
- ● qubit corrected by X gate

☐	$+1$ syndrome
▨	-1 syndrome

We can attempt to correct errors by pairing together -1 syndrome measurements with *shortest paths* of corrections.

Correcting errors



● unaffected qubit

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□ +1 syndrome

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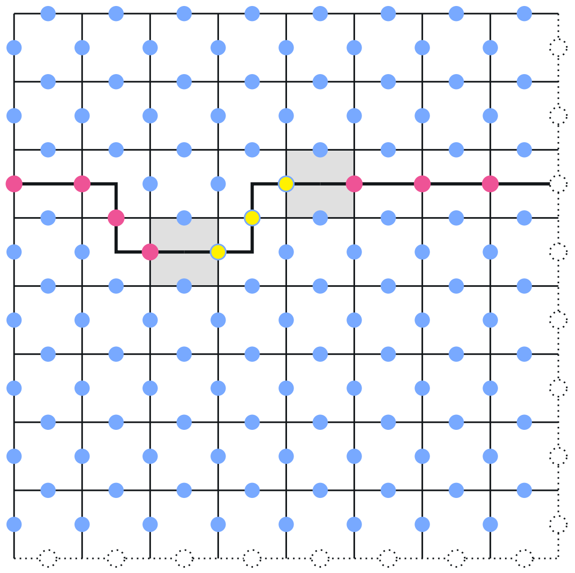
Correcting errors



● unaffected qubit

● qubit affected by X error

● qubit corrected by X gate

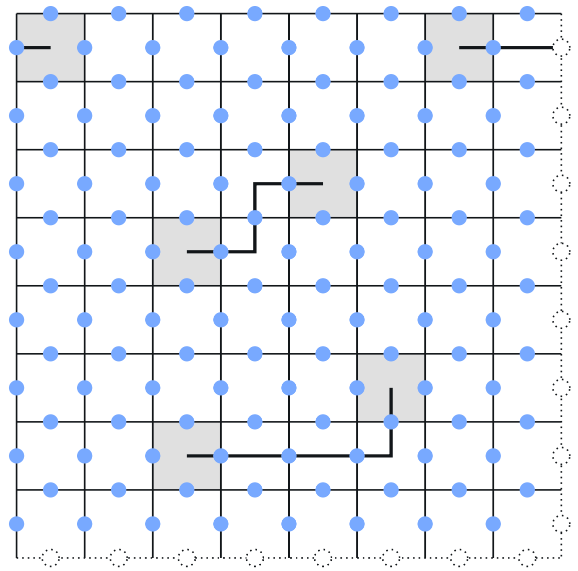
□ +1 syndrome

▨ −1 syndrome

This strategy corrects low-weight errors, but may not work for high-weight errors.

We can attempt to correct errors by pairing together -1 syndrome measurements with *shortest paths* of corrections.

Correcting errors



- ● unaffected qubit
- ● qubit affected by X error
- ● qubit corrected by X gate

□ +1 syndrome

▨ −1 syndrome

This strategy corrects low-weight errors, but may not work for high-weight errors.

Lowest-weight pairings can be found by efficient classical algorithms.

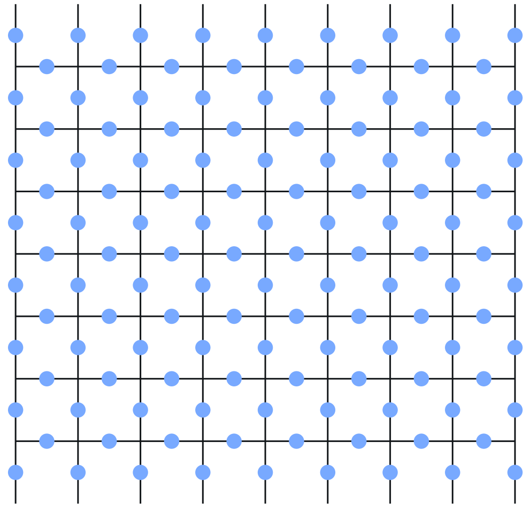
We can attempt to correct errors by pairing together −1 syndrome measurements with *shortest paths* of corrections.

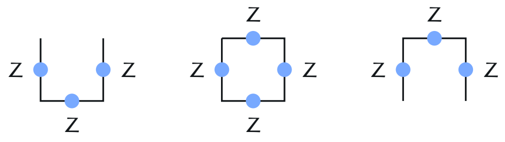
Depending on the noise model, lowest-weight pairings may not correct the most likely errors — but the method works well for simple noise models.

Other code families

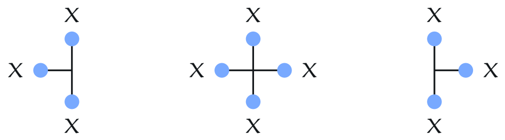
Surface codes

The toric code doesn't actually require periodic boundaries — it can be defined on a *two-dimensional surface* instead.





Z stabilizer generators



X stabilizer generators

Other code families

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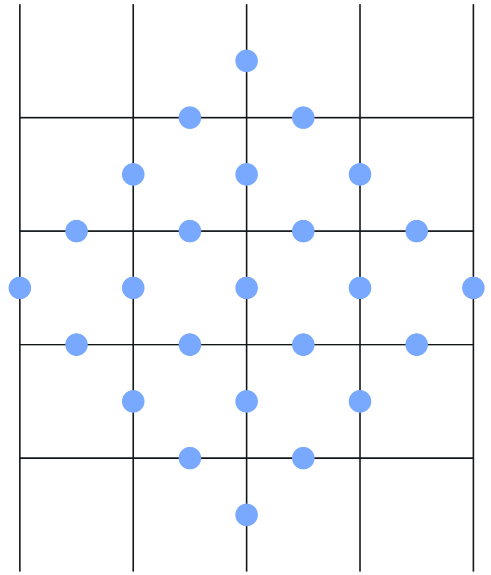

Other code families

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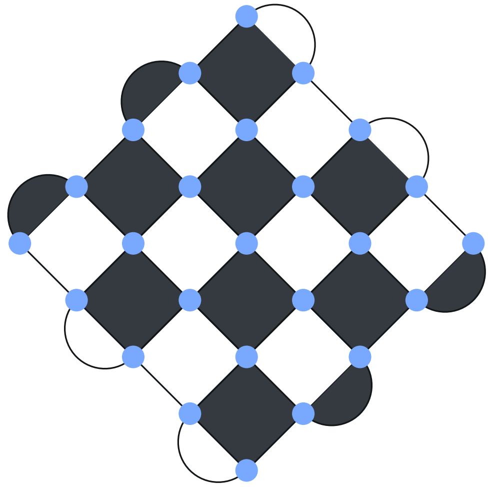

◆ X stabilizer generator

◇ Z stabilizer generator

Other code families

Surface codes

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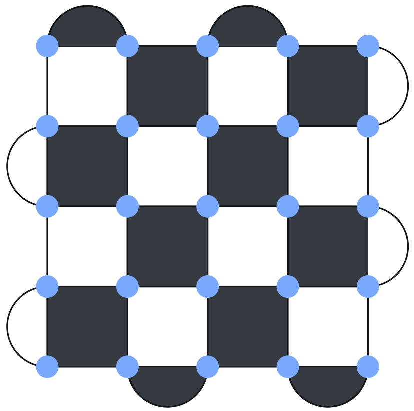


■ X stabilizer generator

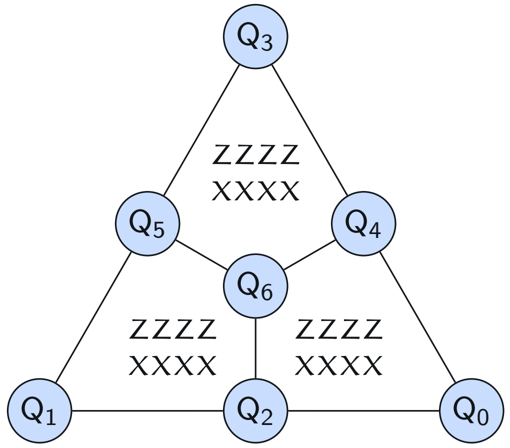
□ Z stabilizer generator

Other code families

Color codes

Consider the 7-qubit Steane code for qubits $(Q_6, Q_5, Q_4, Q_3, Q_2, Q_1, Q_0)$.

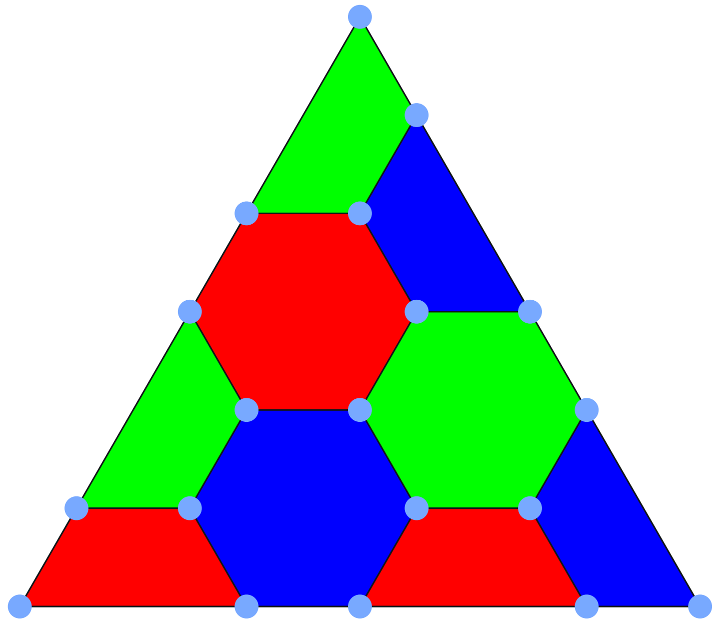
$$\begin{array}{ccccccc}
Z & Z & Z & Z & \mathbb{1} & \mathbb{1} & \mathbb{1} \\
Z & Z & \mathbb{1} & \mathbb{1} & Z & Z & \mathbb{1} \\
Z & \mathbb{1} & Z & \mathbb{1} & Z & \mathbb{1} & Z \\
X & X & X & X & \mathbb{1} & \mathbb{1} & \mathbb{1} \\
X & X & \mathbb{1} & \mathbb{1} & X & X & \mathbb{1} \\
X & \mathbb{1} & X & \mathbb{1} & X & \mathbb{1} & X
\end{array}$$



Color codes generalize this basic pattern to other graphs and lattices.

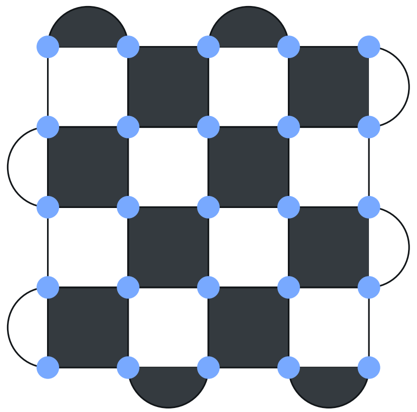
Other code families

Color codes

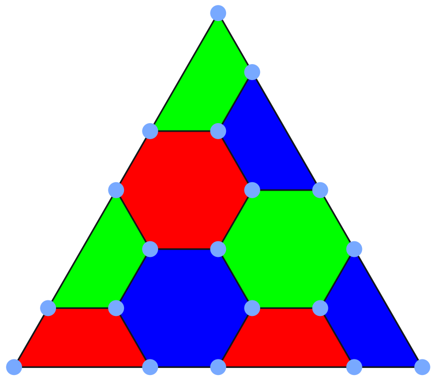


Other code families

Surface codes



Color codes



Many other constructions for quantum error correcting codes are known.

Example: Gross code

The *gross code* is a recently discovered $[[144, 12, 12]]$ stabilizer code.

It requires an additional 144 qubits for performing syndrome measurements and has a biplanar embedding.