

Understanding Quantum Information and Computation

Lesson 14

The Stabilizer Formalism

John Watrous

Pauli operations

$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Anti-commutation relations:

$$XY = -YX \quad XZ = -ZX \quad YZ = -ZY$$

Multiplication rules:

$$XY = iZ \quad YZ = iX \quad ZX = iY \quad XX = YY = ZZ = \mathbb{1}$$

An ***n-qubit Pauli operation*** is the n -fold tensor product of Pauli matrices. Its ***weight*** is the number of non-identity Pauli matrices in the tensor product.

$$\mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \quad \leftarrow \text{weight } 0$$

$$X \otimes X \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \quad \leftarrow \text{weight } 2$$

$$X \otimes Y \otimes Z \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes X \otimes Y \otimes Z \quad \leftarrow \text{weight } 6$$

Pauli operations as generators

$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Suppose that P_1, \dots, P_r are n -qubit Pauli operations.

The set **generated** by P_1, \dots, P_r includes all matrices that can be obtained from P_1, \dots, P_r by multiplication (taking any number of each operation and in any order).

Notation: $\langle P_1, \dots, P_r \rangle$

Example 1

$$\langle X, Y, Z \rangle = \{ \alpha P : \alpha \in \{1, i, -1, -i\}, P \in \{\mathbb{1}, X, Y, Z\} \} \quad (16 \text{ elements})$$

Example 2

$$\langle X, Z \rangle = \{ \mathbb{1}, X, Z, XZ, -\mathbb{1}, -X, -Z, -XZ \} \quad (8 \text{ elements})$$

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Example 2

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Example 3

$$\langle X \otimes X, Z \otimes Z \rangle = \{ \mathbb{1} \otimes \mathbb{1}, X \otimes X, -Y \otimes Y, Z \otimes Z \} \quad (4 \text{ elements})$$

Pauli observables

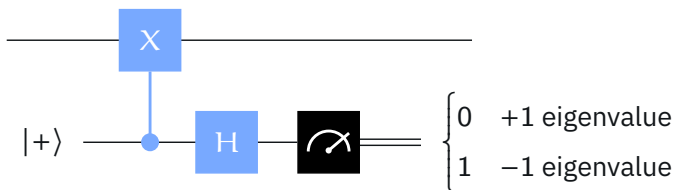
Pauli matrices describe unitary operations — but they also describe *measurements*.

More precisely, we can associate each Pauli matrix with a *projective measurement* defined by its eigenvectors.

$$X = |+\rangle\langle+| - |-\rangle\langle-| \quad Y = |+i\rangle\langle+i| - |-i\rangle\langle-i| \quad Z = |0\rangle\langle 0| - |1\rangle\langle 1|$$

For example, an X measurement is a measurement with respect to the basis $\{|+\rangle, |-\rangle\}$. Equivalently it is the measurement described by the set $\{|+\rangle\langle+|, |-\rangle\langle-|\}$.

We can perform this measurement non-destructively using *phase estimation*.



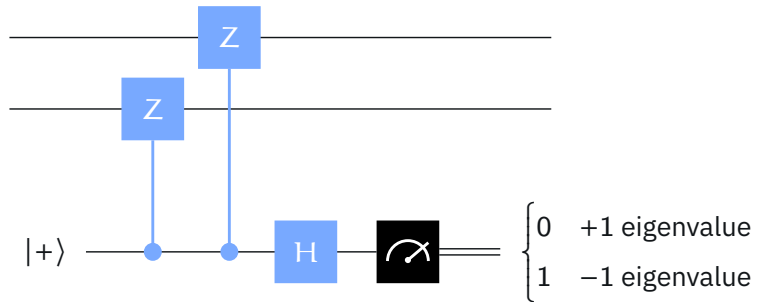
Pauli observables

This extends naturally to n -qubit Pauli operations. For example, consider $Z \otimes Z$.

$$\begin{aligned} Z \otimes Z &= (|0\rangle\langle 0| - |1\rangle\langle 1|) \otimes (|0\rangle\langle 0| - |1\rangle\langle 1|) \\ &= (|00\rangle\langle 00| + |11\rangle\langle 11|) - (|01\rangle\langle 01| + |10\rangle\langle 10|) \end{aligned}$$

The associated measurement is the two-outcome projective measurement described by the set $\{|00\rangle\langle 00| + |11\rangle\langle 11|, |01\rangle\langle 01| + |10\rangle\langle 10|\}$.

Again we can perform this measurement non-destructively using phase estimation.



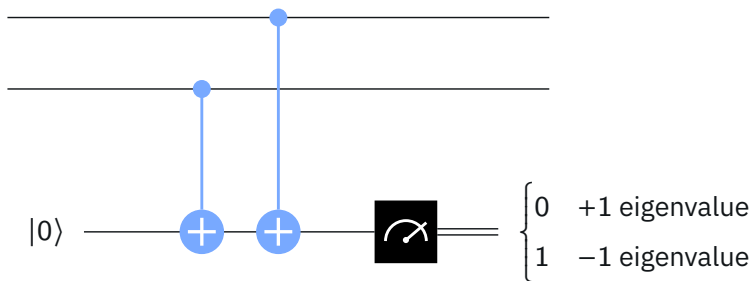
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Repetition code revisited

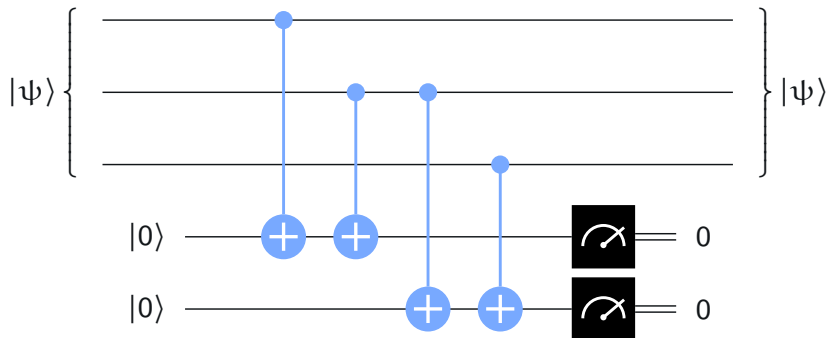
The 3-bit repetition code encodes qubit states as follows:

$$\alpha|0\rangle + \beta|1\rangle \mapsto \alpha|000\rangle + \beta|111\rangle = |\psi\rangle$$

To check that the 3-qubit state $|\psi\rangle$ is a valid encoding of a qubit, it suffices to check these two equations:

$$(Z \otimes Z \otimes \mathbb{1})|\psi\rangle = |\psi\rangle$$

$$(\mathbb{1} \otimes Z \otimes Z)|\psi\rangle = |\psi\rangle$$



Repetition code revisited

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$$\begin{aligned}(Z \otimes Z \otimes \mathbb{1})|\psi\rangle &= |\psi\rangle \\ (\mathbb{1} \otimes Z \otimes Z)|\psi\rangle &= |\psi\rangle\end{aligned}$$

The 3-qubit Pauli operations $Z \otimes Z \otimes \mathbb{1}$ and $\mathbb{1} \otimes Z \otimes Z$ are *stabilizer generators* for this code. The *stabilizer* for the code is the set generated by the stabilizer generators.

$$\langle Z \otimes Z \otimes \mathbb{1}, \mathbb{1} \otimes Z \otimes Z \rangle = \{ \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}, Z \otimes Z \otimes \mathbb{1}, \mathbb{1} \otimes Z \otimes Z, Z \otimes \mathbb{1} \otimes Z \}$$

Bit-flip detection

$$\alpha|0\rangle + \beta|1\rangle \mapsto \alpha|000\rangle + \beta|111\rangle = |\psi\rangle$$

$$(Z \otimes Z \otimes \mathbb{1})|\psi\rangle = |\psi\rangle$$

$$(\mathbb{1} \otimes Z \otimes Z)|\psi\rangle = |\psi\rangle$$

Suppose a bit-flip error occurs on the leftmost qubit.

$$|\psi\rangle \mapsto (X \otimes \mathbb{1} \otimes \mathbb{1})|\psi\rangle$$

By treating the stabilizer generators as observables, we can detect this error.

$$(Z \otimes Z \otimes \mathbb{1})(X \otimes \mathbb{1} \otimes \mathbb{1})|\psi\rangle = -(X \otimes \mathbb{1} \otimes \mathbb{1})(Z \otimes Z \otimes \mathbb{1})|\psi\rangle = -(X \otimes \mathbb{1} \otimes \mathbb{1})|\psi\rangle$$

$$(\mathbb{1} \otimes Z \otimes Z)(X \otimes \mathbb{1} \otimes \mathbb{1})|\psi\rangle = (X \otimes \mathbb{1} \otimes \mathbb{1})(\mathbb{1} \otimes Z \otimes Z)|\psi\rangle = (X \otimes \mathbb{1} \otimes \mathbb{1})|\psi\rangle$$

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Bit-flip detection

$$\alpha|0\rangle + \beta|1\rangle \mapsto \alpha|000\rangle + \beta|111\rangle = |\psi\rangle$$

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$$(\mathbb{1} \otimes Z \otimes Z)(X \otimes \mathbb{1} \otimes \mathbb{1}) = (X \otimes \mathbb{1} \otimes \mathbb{1})(\mathbb{1} \otimes Z \otimes Z)$$

	$\mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}$	$X \otimes \mathbb{1} \otimes \mathbb{1}$	$\mathbb{1} \otimes X \otimes \mathbb{1}$	$\mathbb{1} \otimes \mathbb{1} \otimes X$
$Z \otimes Z \otimes \mathbb{1}$	+1	-1	-1	+1
$\mathbb{1} \otimes Z \otimes Z$	+1	+1	-1	-1

syndromes

Syndromes

	$\mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}$	$X \otimes \mathbb{1} \otimes \mathbb{1}$	$\mathbb{1} \otimes X \otimes \mathbb{1}$	$\mathbb{1} \otimes \mathbb{1} \otimes X$
$Z \otimes Z \otimes \mathbb{1}$	+1	-1	-1	+1
$\mathbb{1} \otimes Z \otimes Z$	+1	+1	-1	-1

syndromes

The syndromes partition the 8-dimensional space into four 2-dimensional subspaces.

		$\mathbb{1} \otimes Z \otimes Z$	
		+1	-1
$Z \otimes Z \otimes \mathbb{1}$	+1	$ 000\rangle$	$ 001\rangle$
		$ 111\rangle$	$ 110\rangle$
	-1	$ 100\rangle$	$ 010\rangle$
		$ 011\rangle$	$ 101\rangle$

They also partition the 3-qubit Pauli operations into 4 equal-size collections. For example, $\mathbb{1} \otimes \mathbb{1} \otimes Z$, $Z \otimes Z \otimes Z$, and $X \otimes X \otimes X$ all cause the same syndrome (+1, +1).

Syndromes

The syndromes partition the 8-dimensional space into four 2-dimensional subspaces.

		$\mathbb{1} \otimes Z \otimes Z$	
		$\overbrace{\hspace{1.5cm}}$	
		+1	-1
$Z \otimes Z \otimes \mathbb{1}$	+1	$ 000\rangle$	$ 001\rangle$
		$ 111\rangle$	$ 110\rangle$
	-1	$ 100\rangle$	$ 010\rangle$
		$ 011\rangle$	$ 101\rangle$

They also partition the 3-qubit Pauli operations into 4 equal-size collections. For example, $\mathbb{1} \otimes \mathbb{1} \otimes Z$, $Z \otimes Z \otimes Z$, and $X \otimes X \otimes X$ all cause the same syndrome (+1, +1).

Pauli operations that commute with every stabilizer generator but are not themselves in the stabilizer act like Pauli operations on the encoded qubit.

Stabilizer codes

A set $\{P_1, \dots, P_r\}$ of n -qubit Pauli operations are stabilizer generators for a *stabilizer code* if these properties are satisfied:

1. The stabilizer generators all *commute* with one another.

$$P_j P_k = P_k P_j \quad (\text{for all } j, k \in \{1, \dots, r\})$$

2. The stabilizer generators form a *minimal generating set*.

$$P_k \notin \langle P_1, \dots, P_{k-1}, P_{k+1}, \dots, P_r \rangle \quad (\text{for each } k \in \{1, \dots, r\})$$

3. At least one nonzero vector is fixed by all of the stabilizer generators.

$$-\mathbb{1}^{\otimes n} \notin \langle P_1, \dots, P_r \rangle$$

The *code space* defined by the stabilizer generators contains all vectors that are fixed by all of the stabilizer generators.

$$\{|\psi\rangle : |\psi\rangle = P_1|\psi\rangle = \dots = P_r|\psi\rangle\}$$

Examples

3-bit repetition code (bit-flips)

$$\begin{array}{c} Z \otimes Z \otimes 1 \\ 1 \otimes Z \otimes Z \end{array}$$

3-bit repetition code (phase-flips)

$$\begin{array}{c} X \otimes X \otimes 1 \\ 1 \otimes X \otimes X \end{array}$$

9-qubit Shor code

$$\begin{array}{c} Z \otimes Z \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \\ 1 \otimes Z \otimes Z \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \\ 1 \otimes 1 \otimes 1 \otimes Z \otimes Z \otimes 1 \otimes 1 \otimes 1 \otimes 1 \\ 1 \otimes 1 \otimes 1 \otimes 1 \otimes Z \otimes Z \otimes 1 \otimes 1 \otimes 1 \\ 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes Z \otimes Z \otimes 1 \\ 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes Z \otimes Z \\ X \otimes X \otimes X \otimes X \otimes X \otimes X \otimes 1 \otimes 1 \otimes 1 \\ 1 \otimes 1 \otimes 1 \otimes X \otimes X \otimes X \otimes X \otimes X \otimes X \end{array}$$

Examples

3-bit repetition code (bit-flips)

Z Z 1
1 Z Z

3-bit repetition code (phase-flips)

X X 1
1 X X

9-qubit Shor code

Z Z 1 1 1 1 1 1 1
1 Z Z 1 1 1 1 1 1
1 1 1 Z Z 1 1 1 1
1 1 1 1 Z Z 1 1 1
1 1 1 1 1 1 Z Z 1
1 1 1 1 1 1 1 Z Z
X X X X X X 1 1 1
1 1 1 X X X X X X

Examples

7-qubit Steane code

```
Z Z Z Z 1 1 1
Z Z 1 1 Z Z 1
Z 1 Z 1 Z 1 Z
X X X X 1 1 1
X X 1 1 X X 1
X 1 X 1 X 1 X
```

5-qubit code

```
X Z Z X 1
1 X Z Z X
X 1 X Z Z
Z X 1 X Z
```

E-bit stabilizer code

```
Z Z
X X
```

GHZ stabilizer code

```
Z Z 1
1 Z Z
X X X
```

Code space dimension

Suppose that $\{P_1, \dots, P_r\}$ are n -qubit stabilizer generators for a stabilizer code.

1. $P_j P_k = P_k P_j$ for all $j, k \in \{1, \dots, r\}$
2. $P_k \notin \langle P_1, \dots, P_{k-1}, P_{k+1}, \dots, P_r \rangle$ for each $k \in \{1, \dots, r\}$
3. $-\mathbb{1} \notin \langle P_1, \dots, P_r \rangle$

Theorem

The code space defined by $\{P_1, \dots, P_r\}$ has dimension 2^{n-r} .

(Equivalently, the code defined by these generators encodes $n - r$ qubits.)

3-bit repetition code (bit-flips)

$Z Z \mathbb{1}$

$\mathbb{1} Z Z$

$n = 3$ qubits

$r = 2$ stabilizer generators

$\Rightarrow 3 - 2 = 1$ encoded qubit

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5-qubit code

$X Z Z X \mathbb{1}$
 $\mathbb{1} X Z Z X$
 $X \mathbb{1} X Z Z$
 $Z X \mathbb{1} X Z$

$n = 5$ qubits

$r = 4$ stabilizer generators

$\Rightarrow 5 - 4 = 1$ encoded qubit

Code space dimension

Suppose that $\{P_1, \dots, P_r\}$ are n -qubit stabilizer generators for a stabilizer code.

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E-bit stabilizer code

Z Z

X X

$n = 2$ qubits

$r = 2$ stabilizer generators

$\Rightarrow 2 - 2 = 0$ encoded qubits

The code space is the 1-dimensional space spanned by the vector $|\phi^+\rangle$.

Code space dimension

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2. $P_k \notin \langle P_1, \dots, P_{k-1}, P_{k+1}, \dots, P_r \rangle$ for each $k \in \{1, \dots, r\}$
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Theorem

The code space defined by $\{P_1, \dots, P_r\}$ has dimension 2^{n-r} .

(Equivalently, the code defined by these generators encodes $n - r$ qubits.)

Every element in the stabilizer can be written in a unique way as $P_1^{a_1} \dots P_r^{a_r}$ for $a_1, \dots, a_r \in \{0, 1\}$.

$$P_1^{a_1} \dots P_r^{a_r} = \mathbb{1}^{\otimes n} \iff (a_1, \dots, a_r) = (0, \dots, 0)$$

The projection Π_k onto the $+1$ eigenspace of P_k can be expressed like this:

$$\Pi_k = \frac{\mathbb{1}^{\otimes n} + P_k}{2}$$

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The projections Π_1, \dots, Π_r commute. The projection onto the code space is their product.

$$\Pi_1 \dots \Pi_r = \left(\frac{\mathbb{1}^{\otimes n} + P_1}{2} \right) \dots \left(\frac{\mathbb{1}^{\otimes n} + P_r}{2} \right) = \frac{1}{2^r} \sum_{a_1, \dots, a_r \in \{0, 1\}} P_1^{a_1} \dots P_r^{a_r}$$

Code space dimension

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The dimension of the code space is the trace of this projection.

$$\text{Tr}(\Pi_1 \dots \Pi_r) = \frac{1}{2^r} \sum_{a_1, \dots, a_r \in \{0, 1\}} \text{Tr}(P_1^{a_1} \dots P_r^{a_r}) = \frac{2^n}{2^r} = 2^{n-r}$$

Clifford operations and encodings

Clifford operations

Clifford operations are unitary operations that can be implemented by quantum circuits with gates from this list:

- Hadamard gates
- S gates
- CNOT gates

Up to a global phase, an n -qubit unitary operation is a Clifford operation if and only if it maps n -qubit Pauli operations to n -qubit Pauli operations by conjugation.

Equivalently, U is a Clifford operation (up to a global phase) if for every $P_0, \dots, P_{n-1} \in \{\mathbb{1}, X, Y, Z\}$ there exist $Q_0, \dots, Q_{n-1} \in \{\mathbb{1}, X, Y, Z\}$ such that

$$U(P_{n-1} \otimes \dots \otimes P_0)U^\dagger = \pm Q_{n-1} \otimes \dots \otimes Q_0$$

Clifford operations are *not universal* for quantum computation.

There are only finitely many n -qubit Clifford operations and their actions on standard basis states can be efficiently simulated classically by the *Gottesman-Knill theorem*.

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Encodings for stabilizer codes can always be performed by Clifford operations. At most $O(n^2 / \log(n))$ gates are required.

Clifford operations and encodings

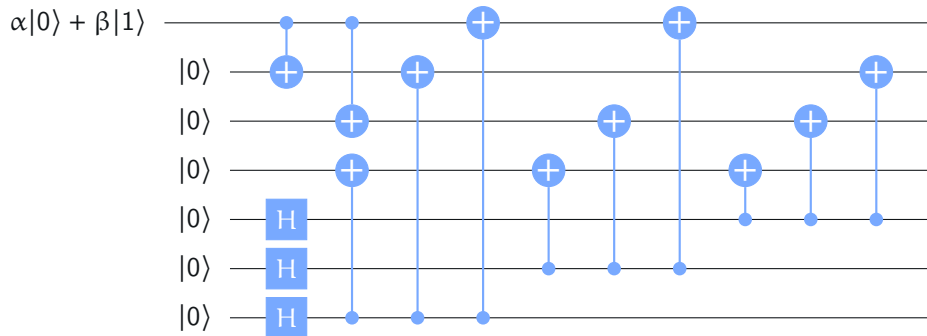
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Example: encoder for the 7-qubit Steane code



Detecting errors

Let P_1, \dots, P_r be stabilizer generators for an n -qubit stabilizer code, and let E be an n -qubit Pauli operation, representing a *hypothetical error*.

Errors are detected in a stabilizer code by *measuring the stabilizer generators* (as observables). The r outcomes form the syndrome.

Case 1: $E = \alpha Q$ for $Q \in \langle P_1, \dots, P_r \rangle$.

This error *does nothing* to vectors in the code space: $E|\psi\rangle = \alpha|\psi\rangle$ for every encoded state $|\psi\rangle$.

Case 2: $E \neq \alpha Q$ for $Q \in \langle P_1, \dots, P_r \rangle$, but $EP_k = P_k E$ for every $k \in \{1, \dots, r\}$.

This error changes vectors in the code space and goes *undetected* by the code.

Case 3: $P_k E = -E P_k$ for at least one $k \in \{1, \dots, r\}$.

This error is *detected* by the code.

The *distance* of a stabilizer code is the *minimum weight* of a Pauli operation that changes vectors in the code space but goes undetected by the code.

Notation: an $[[n, m, d]]$ stabilizer code is one that encodes m qubits into n qubits and has distance d .

7-qubit Steane code

```
Z Z Z Z 1 1 1
Z Z 1 1 Z Z 1
Z 1 Z 1 Z 1 Z
X X X X 1 1 1
X X 1 1 X X 1
X 1 X 1 X 1 X
```

The *distance* is the minimum weight of an n -qubit Pauli operation that

1. commutes with every stabilizer generator, and
2. is not proportional to a stabilizer element.

This code has distance 3.

We can first reason that every Pauli operation with weight at most 2 that commutes with every stabilizer generator must be the identity operation.

```
P Q 1 1 1 1 1
Z 1 Z 1 Z 1 Z
X 1 X 1 X 1 X
```

7-qubit Steane code

```
Z Z Z Z 1 1 1
Z Z 1 1 Z Z 1
Z 1 Z 1 Z 1 Z
X X X X 1 1 1
X X 1 1 X X 1
X 1 X 1 X 1 X
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1 Q 1 1 1 1 1
Z 1 Z 1 Z 1 Z
X 1 X 1 X 1 X
Z Z Z Z 1 1 1
X X X X 1 1 1
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7-qubit Steane code

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Z Z Z Z 1 1 1
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1 1 1 1 1 1 1
Z 1 Z 1 Z 1 Z
X 1 X 1 X 1 X
Z Z Z Z 1 1 1
X X X X 1 1 1
```

7-qubit Steane code

```
Z Z Z Z 1 1 1
Z Z 1 1 Z Z 1
Z 1 Z 1 Z 1 Z
X X X X 1 1 1
X X 1 1 X X 1
X 1 X 1 X 1 X
```

The *distance* is the minimum weight of an n -qubit Pauli operation that

1. commutes with every stabilizer generator, and
2. is not proportional to a stabilizer element.

This code has distance 3. ✓

We can first reason that every Pauli operation with weight at most 2 that commutes with every stabilizer generator must be the identity operation.

On the other hand, there are weight 3 Pauli operations that commute with every stabilizer generator and fall outside of the stabilizer.

Two examples:

```
1 1 1 1 X X X
1 1 1 1 Z Z Z
```

Correcting errors

Let P_1, \dots, P_r be stabilizer generators for an n -qubit stabilizer code.

- The 2^r syndromes partition the n -qubit Pauli operations into equal-size sets, with $4^n/2^r$ Pauli operations in each set.
- If E is an error and $S \in \langle P_1, \dots, P_r \rangle$ is a stabilizer element, then E and ES are equivalent errors: $E|\psi\rangle = ES|\psi\rangle$ for every $|\psi\rangle$ in the code space.
- This leaves 4^{n-r} inequivalent classes of errors for each syndrome.

So, unless $r = n$ (i.e., the code space is one-dimensional) we cannot correct every error. Rather, we must choose **one correction operation for each syndrome** (which corrects at most one class of equivalent errors).

Natural strategy

For each syndrome s , choose a **lowest weight** Pauli operation that causes the syndrome s as the corresponding correction operation.

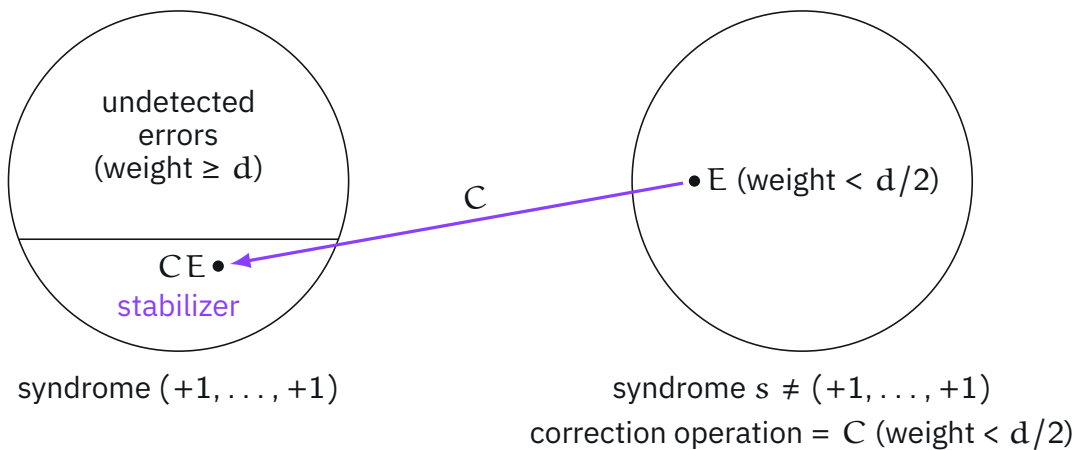
For a distance d stabilizer code, this strategy corrects all errors having weight at most $(d-1)/2$.

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Unfortunately, for a given choice of stabilizer generators and a syndrome, it is *computationally difficult* to find the lowest weight Pauli operation causing that syndrome.

Finding codes for which this can be done efficiently is part of the artistry in code design.