Understanding Quantum Information and Computation

Lesson 11

General Measurements

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Descriptions of measurements

Measurements represent an interface between quantum and classical information:

- Performing a measurement on a system extracts classical information about its quantum state.
- In general, the system is changed (or destroyed) in the process.

Initially our focus will be on *destructive measurements* — which produce a classical outcome alone. (The post-measurement state of the system is not specified.)

Two ways to describe destructive measurements

1. As *collections of matrices,* one for each measurement outcome.
2. As *channels* whose outputs are always classical states (represented by diagonal density matrices).

Non-destructive measurements will be discussed later in the lesson. (They can always be described as compositions of destructive measurements and channels.)

Measurements as matrices

Suppose X is a system to be measured. For simplicity we will assume the following:

- The classical state set of X is $\{0, \ldots, n-1\}$.
- The set of measurement outcomes is $\{0, \ldots, m-1\}$.

Recollection: projective measurements

A *projective measurement* is described by a collection of projection matrices $\{\Pi_0, \ldots, \Pi_{m-1}\}$ satisfying this condition:

$$\Pi_0 + \cdots + \Pi_{m-1} = \mathbb{1}_X$$

If the state of X is ρ, each outcome a appears with this probability:

$$\mathrm{Tr}(\Pi_a \rho)$$

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General measurements

A *general measurement* is described by a collection of positive semidefinite matrices $\{P_0, \ldots, P_{m-1}\}$ satisfying this condition:

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We necessarily obtain a probability vector $(\operatorname{Tr}(P_0\rho), \ldots, \operatorname{Tr}(P_{m-1}\rho))$:

- These are nonnegative real numbers: $Q, R \geq 0 \Rightarrow \operatorname{Tr}(QR) \geq 0$.
- These numbers sum to 1:

$$\operatorname{Tr}(P_0\rho) + \cdots + \operatorname{Tr}(P_{m-1}\rho) = \operatorname{Tr}\big((P_0 + \cdots + P_{m-1})\rho\big) = \operatorname{Tr}(\rho) = 1$$

Examples

Projections are always positive semidefinite, so every projective measurement is an example of a general measurement.

Example 1

A standard basis measurement of a qubit can be represented by $\{P_0, P_1\}$ where

$$P_0 = |0\rangle\langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad P_1 = |1\rangle\langle 1| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Measuring a qubit in the state ρ results in outcome probabilities as follows.

$$\begin{aligned} \text{Prob}(\text{outcome} = 0) &= \text{Tr}(P_0 \rho) = \text{Tr}\big(|0\rangle\langle 0|\rho\big) = \langle 0|\rho|0\rangle \\ \text{Prob}(\text{outcome} = 1) &= \text{Tr}(P_1 \rho) = \text{Tr}\big(|1\rangle\langle 1|\rho\big) = \langle 1|\rho|1\rangle \end{aligned}$$

Examples

Example 2

Define P_0 and P_1 as follows.

$$P_0 = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix} \qquad P_1 = \begin{pmatrix} \frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

Measuring a qubit in the $|+\rangle$ state results in outcome probabilities as follows.

$$\text{Prob}(\text{outcome} = 0) = \text{Tr}(P_0 |+\rangle\langle+|) = \text{Tr}\left(\begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\right) = \frac{5}{6}$$

$$\text{Prob}(\text{outcome} = 1) = \text{Tr}(P_1 |+\rangle\langle+|) = \text{Tr}\left(\begin{pmatrix} \frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\right) = \frac{1}{6}$$

Examples

Example 3

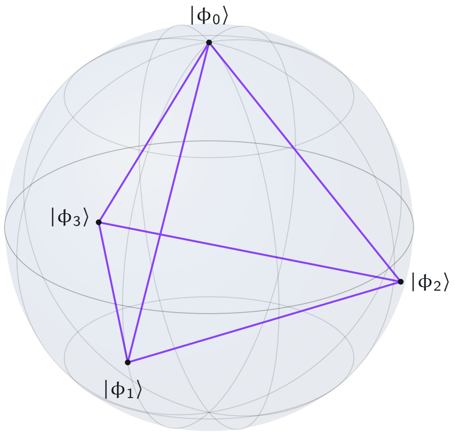
The *tetrahedral states* are defined as follows.

$$|\phi_0\rangle = |0\rangle$$

$$|\phi_1\rangle = \frac{1}{\sqrt{3}}|0\rangle + \sqrt{\frac{2}{3}}|1\rangle$$

$$|\phi_2\rangle = \frac{1}{\sqrt{3}}|0\rangle + \sqrt{\frac{2}{3}}e^{2\pi i/3}|1\rangle$$

$$|\phi_3\rangle = \frac{1}{\sqrt{3}}|0\rangle + \sqrt{\frac{2}{3}}e^{-2\pi i/3}|1\rangle$$



We can define a measurement $\{P_0, P_1, P_2, P_3\}$ as follows.

$$P_0 = \frac{|\phi_0\rangle\langle\phi_0|}{2} \qquad P_1 = \frac{|\phi_1\rangle\langle\phi_1|}{2} \qquad P_2 = \frac{|\phi_2\rangle\langle\phi_2|}{2} \qquad P_3 = \frac{|\phi_3\rangle\langle\phi_3|}{2}$$

Measurements as channels

Recall that classical (probabilistic) states can be represented by diagonal density matrices.

Any general measurement can be described as a *channel* Φ:

- The input system X is the system being measured.
- The classical states of the output system Y are the possible measurement outcomes $\{0, \ldots, m-1\}$.
- For every input state ρ of X, the output state $\Phi(\rho)$ is a diagonal density matrix.

Example: standard basis measurement

The *completely dephasing* channel Δ describes a standard basis measurement of a qubit:

$$\Delta(\rho) = \langle 0|\rho|0\rangle\, |0\rangle\langle 0| + \langle 1|\rho|1\rangle\, |1\rangle\langle 1|$$

Measurements as channels

Recall that classical (probabilistic) states can be represented by diagonal density matrices.

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- The input system X is the system being measured.
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- For every input state ρ of X, the output state $\Phi(\rho)$ is a diagonal density matrix.

Equivalence to matrix description

A channel Φ from X to Y has the property that $\Phi(\rho)$ is always diagonal if and only if

$$\Phi(\rho) = \sum_{a=0}^{m-1} \mathrm{Tr}(P_a \rho)\, |a\rangle\langle a|$$

for a measurement $\{P_0, \ldots, P_{m-1}\}$.

Partial measurements

Suppose that a pair of systems (X, Z) is in a state ρ and a measurement $\{P_0, \ldots, P_{m-1}\}$ is performed on X.

This results in a measurement outcome — and in addition the state of Z may change depending on the outcome.

Outcome probabilities

The probabilities for different measurement outcome probabilities to appear depend only on the measurement and the *reduced state* ρ_{X} of X.

$$\text{Prob}(\text{outcome} = a) = \text{Tr}(P_a \rho_{\mathsf{X}}) = \text{Tr}\big(P_a \text{Tr}_{\mathsf{Z}}(\rho)\big) = \text{Tr}\big((P_a \otimes \mathbb{I}_{\mathsf{Z}})\rho\big)$$

Partial measurements

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States conditioned on measurement outcomes

To determine the state of Z conditioned on a given measurement outcome we can turn to the channel description of the measurement:

$$\Phi(\sigma) = \sum_{a=0}^{m-1} \operatorname{Tr}(P_a \sigma) \, |a\rangle\langle a|$$

Applying this channel to X results in this state:

$$\sum_{a=0}^{m-1} |a\rangle\langle a| \otimes \operatorname{Tr}_{\mathsf{X}}\bigl((P_a \otimes \mathbb{I}_{\mathsf{Z}})\rho\bigr)$$

The state of Z *conditioned* on the outcome a can be obtained by normalizing the matrix $\operatorname{Tr}_{\mathsf{X}}\bigl((P_a \otimes \mathbb{I}_{\mathsf{Z}})\rho\bigr)$.

Partial measurements

Suppose that a pair of systems (X, Z) is in a state ρ and a measurement $\{P_0, \ldots, P_{m-1}\}$ is performed on X.

This results in a measurement outcome — and in addition the state of Z may change depending on the outcome.

Summary

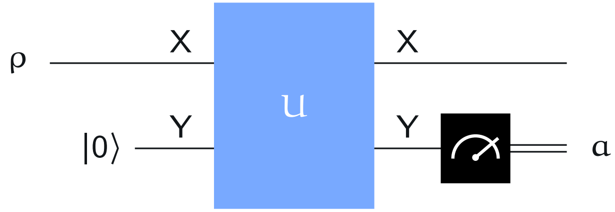
When a measurement $\{P_0, \ldots, P_{m-1}\}$ is performed on X when (X, Z) is in the state ρ, the following happens:

1. Each outcome a appears with probability $\operatorname{Tr}\bigl((P_a \otimes \mathbb{I}_{\mathsf{Z}})\rho\bigr)$.
2. Conditioned on obtaining the outcome a, the state of Z becomes

$$\frac{\operatorname{Tr}_{\mathsf{X}}\bigl((P_a \otimes \mathbb{I}_{\mathsf{Z}})\rho\bigr)}{\operatorname{Tr}\bigl((P_a \otimes \mathbb{I}_{\mathsf{Z}})\rho\bigr)}$$

Naimark's theorem

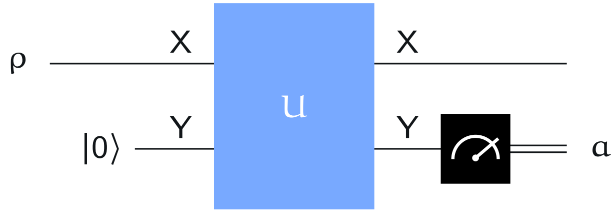
Naimark's theorem is a fundamental fact concerning measurements. It states that every general measurement can be implemented in the following way:



That is, a given general measurement $\{P_0, \ldots, P_{m-1}\}$ on X can be implemented as follows.

1. Introduce an initialized workspace system Y having classical states $\{0, \ldots, m-1\}$.
2. Perform a unitary operation U on the pair (Y, X).
3. Perform a standard basis measurement on Y.

Proof of Naimark's theorem



Naimark's theorem is not difficult to prove… we just need to make a good choice for U and verify that it works.

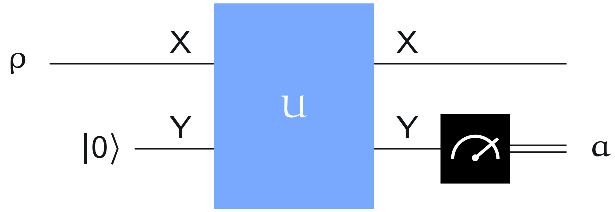
Fact

For every positive semidefinite matrix P, there is a unique positive semidefinite matrix Q such that $Q^2 = P$. This matrix is denoted $\sqrt{P}$.

We can calculate $\sqrt{P}$ using a spectral decomposition of P:

$$P = \sum_{k=0}^{n-1} \lambda_k |\psi_k\rangle\langle\psi_k| \quad \Rightarrow \quad \sqrt{P} = \sum_{k=0}^{n-1} \sqrt{\lambda_k}\, |\psi_k\rangle\langle\psi_k|$$

Proof of Naimark's theorem



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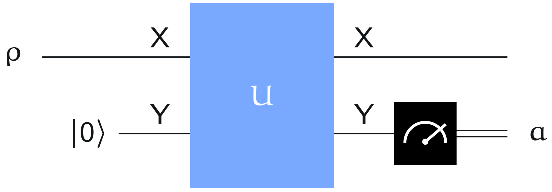
Any unitary matrix U that follows this pattern will work:

$$U = \begin{pmatrix} \sqrt{P_0} & \\ \sqrt{P_1} & \\ \vdots & ? \\ \sqrt{P_{m-1}} & \end{pmatrix}$$

Proof of Naimark's theorem



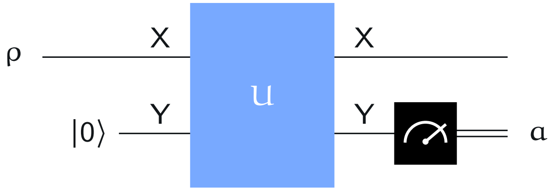
$$U = \begin{pmatrix} \sqrt{P_0} & \\ \sqrt{P_1} & \\ \vdots & ? \\ \sqrt{P_{m-1}} & \end{pmatrix}$$

We need to check two things: (1) that such a matrix U works correctly, and (2) that U can be made unitary.

$$U(|0\rangle\langle 0| \otimes \rho)U^\dagger$$

$$= \begin{pmatrix} \sqrt{P_0} & \\ \sqrt{P_1} & \\ \vdots & ? \\ \sqrt{P_{m-1}} & \end{pmatrix} \begin{pmatrix} \rho & \\ & 0 \end{pmatrix} \begin{pmatrix} \sqrt{P_0} \; \sqrt{P_1} \; \cdots \; \sqrt{P_{m-1}} \\ ? \end{pmatrix}$$

Proof of Naimark's theorem



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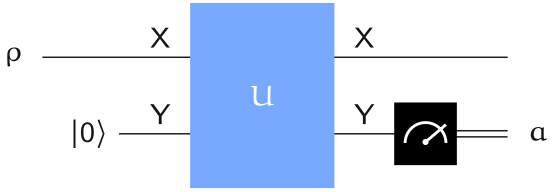
$$U(|0\rangle\langle 0| \otimes \rho)U^\dagger$$

$$= \begin{pmatrix} \sqrt{P_0}\rho\sqrt{P_0} & \cdots & \sqrt{P_0}\rho\sqrt{P_{m-1}} \\ \vdots & \ddots & \vdots \\ \sqrt{P_{m-1}}\rho\sqrt{P_0} & \cdots & \sqrt{P_{m-1}}\rho\sqrt{P_{m-1}} \end{pmatrix}$$

$$= \sum_{a,b=0}^{m-1} |a\rangle\langle b| \otimes \sqrt{P_a}\rho\sqrt{P_b}$$

Proof of Naimark's theorem



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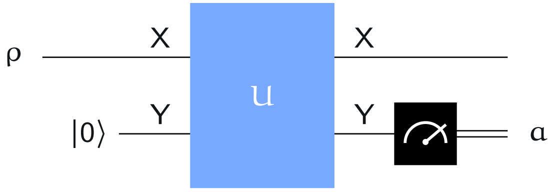
$$\sigma = U(|0\rangle\langle 0| \otimes \rho)U^\dagger = \sum_{a,b=0}^{m-1} |a\rangle\langle b| \otimes \sqrt{P_a}\rho\sqrt{P_b}$$

$$\sigma_Y = \sum_{a,b=0}^{m-1} \operatorname{Tr}\left(\sqrt{P_a}\rho\sqrt{P_b}\right)|a\rangle\langle b|$$

$$\operatorname{Prob}(\text{outcome} = a) = \langle a|\sigma_Y|a\rangle = \operatorname{Tr}\left(\sqrt{P_a}\rho\sqrt{P_a}\right) = \operatorname{Tr}(P_a\rho)$$

Proof of Naimark's theorem



$$U = \begin{pmatrix} \sqrt{P_0} & \\ \sqrt{P_1} & \\ \vdots & ? \\ \sqrt{P_{m-1}} & \end{pmatrix}$$

We need to check two things: (1) that such a matrix U works correctly, and (2) that U can be made unitary.

That U can be made unitary follows from the fact that its first n columns are orthonormal. Denote these first n columns by $|\gamma_0\rangle, \ldots, |\gamma_{n-1}\rangle$.

$$|\gamma_c\rangle = \sum_{a=0}^{m-1} |a\rangle \otimes \sqrt{P_a}|c\rangle \qquad \langle\gamma_c|\gamma_d\rangle = \langle c|\left(\sum_{a=0}^{m-1} \sqrt{P_a}\sqrt{P_a}\right)|d\rangle = \langle c|d\rangle$$

Non-destructive measurements

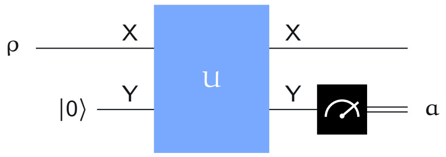
Non-destructive measurements have not only a classical measurement outcome, but also a *post-measurement quantum state* of the system that was measured.

There are different specific ways to formulate them in mathematical terms.

Non-destructive measurements from Naimark's theorem

Consider a general (destructive) measurement $\{P_0, \ldots, P_{m-1}\}$ of a system X.

We can define a *non-destructive* measurement with the same outcome probabilities using Naimark's theorem.



$$U = \begin{pmatrix} \sqrt{P_0} & \\ \sqrt{P_1} & \\ \vdots & ? \\ \sqrt{P_{m-1}} & \end{pmatrix}$$

Conditioned on the outcome a the state of X becomes this:

$$\frac{\sqrt{P_a}\,\rho\,\sqrt{P_a}}{\mathrm{Tr}(P_a \rho)}$$

Non-destructive measurements

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There are different specific ways to formulate them in mathematical terms.

Non-destructive measurements from Kraus matrices

Suppose $M_0, \ldots, M_{m-1}$ are square matrices satisfying this equation:

$$\sum_{a=0}^{m-1} M_a^\dagger M_a = \mathbb{1}$$

They specify a non-destructive measurement. For a system in the state ρ:

$$\Pr(\text{outcome} = a) = \operatorname{Tr}\left(M_a \rho M_a^\dagger\right) = \operatorname{Tr}\left(M_a^\dagger M_a \rho\right)$$

Conditioned on the outcome a the state of the measured system becomes this:

$$\frac{M_a \rho M_a^\dagger}{\operatorname{Tr}\left(M_a \rho M_a^\dagger\right)}$$

State discrimination & tomography

Quantum state discrimination

Let $\rho_0, \ldots, \rho_{m-1}$ be quantum states of a system X and let $(p_0, \ldots, p_{m-1})$ be a probability vector.

- An element $a \in \{0, \ldots, m-1\}$ is chosen at random according to the probabilities $(p_0, \ldots, p_{m-1})$.
- The system X is prepared in the state ρ_a.
- Goal: determine a by measuring X.

Quantum state tomography

Let ρ be an unknown quantum states of a system.

- Identical systems $X_1, \ldots, X_N$ are each independently prepared in the state ρ.
- Goal: approximate ρ by measuring $X_1, \ldots, X_N$.

Discriminating pairs of states

Quantum state discrimination

Let $\rho_0, \ldots, \rho_{m-1}$ be quantum states of a system X and let $(p_0, \ldots, p_{m-1})$ be a probability vector.

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When $m = 2$ for state discrimination, the goal is to distinguish between a *pair of states.*

Pairs of states are optimally discriminated by the *Helstrom measurement.*

Discriminating pairs of states

Pairs of states are optimally discriminated by the *Helstrom measurement.*

This is the projective measurement $\{\Pi_0, \Pi_1\}$ defined as follows.

$$p_0\rho_0 - p_1\rho_1 = \sum_{k=0}^{n-1} \lambda_k |\psi_k\rangle\langle\psi_k|$$

$$S_0 = \{k \in \{0,\ldots,n-1\} : \lambda_k \geq 0\}$$
$$S_1 = \{k \in \{0,\ldots,n-1\} : \lambda_k < 0\}$$

$$\Pi_0 = \sum_{k\in S_0} |\psi_k\rangle\langle\psi_k| \quad \text{and} \quad \Pi_1 = \sum_{k\in S_1} |\psi_k\rangle\langle\psi_k|$$

$$\Pr(\text{correct identification}) = \frac{1}{2} + \frac{1}{2}\sum_{k=0}^{n-1}|\lambda_k| = \frac{1}{2} + \frac{1}{2}\big\|p_0\rho_0 - p_1\rho_1\big\|_1$$

The fact that this is optimal is known as the *Helstrom–Holevo theorem.*

Discriminating 3 or more states

Quantum state discrimination

Let $\rho_0, \ldots, \rho_{m-1}$ be quantum states of a system X and let $(p_0, \ldots, p_{m-1})$ be a probability vector.

- An element $a \in \{0, \ldots, m-1\}$ is chosen at random according to the probabilities $(p_0, \ldots, p_{m-1})$.
- The system X is prepared in the state ρ_a.
- Goal: determine a by measuring X.

When $m \geq 3$ states are to be discriminated, there is *no known formula* for an optimal measurement.

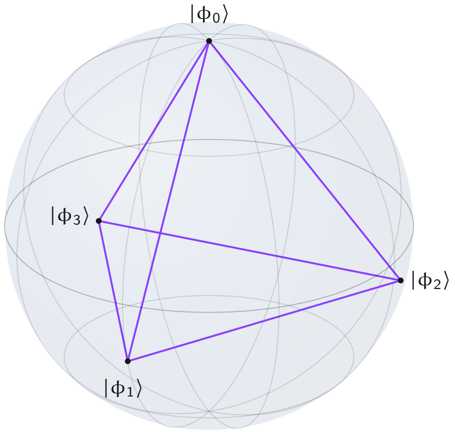
- An optimal measurement can be approximated using *semidefinite programming.*
- The *Holevo-Yuen-Kennedy-Lax* conditions allow a given measurement to be checked for optimality.

Discriminating 3 or more states

Example

Recall that the *tetrahedral states* are defined as follows.

$$|\phi_0\rangle = |0\rangle$$
$$|\phi_1\rangle = \frac{1}{\sqrt{3}}|0\rangle + \sqrt{\frac{2}{3}}|1\rangle$$
$$|\phi_2\rangle = \frac{1}{\sqrt{3}}|0\rangle + \sqrt{\frac{2}{3}}e^{2\pi i/3}|1\rangle$$
$$|\phi_3\rangle = \frac{1}{\sqrt{3}}|0\rangle + \sqrt{\frac{2}{3}}e^{-2\pi i/3}|1\rangle$$



The measurement $\{P_0, P_1, P_2, P_3\}$ discriminates these four states with minimum error.

$$P_0 = \frac{|\phi_0\rangle\langle\phi_0|}{2} \quad P_1 = \frac{|\phi_1\rangle\langle\phi_1|}{2} \quad P_2 = \frac{|\phi_2\rangle\langle\phi_2|}{2} \quad P_3 = \frac{|\phi_3\rangle\langle\phi_3|}{2}$$

Quantum state tomography

Quantum state tomography

Let ρ be an unknown quantum states of a system.

- Identical systems $X_1, \ldots, X_N$ are each independently prepared in the state ρ.
- Goal: approximate ρ by measuring $X_1, \ldots, X_N$.

Different variants of quantum state tomography are considered:

- Measurements can be local (each $X_1, \ldots, X_N$ is measured separately) or global.
- Multiple strategies may be used to find a description of ρ from measurement data.

Qubit tomography

Suppose ρ is an unknown qubit state and $X_1, \ldots, X_N$ are qubits independently prepared in the state ρ. Quantum state tomography can be performed as follows.

1. Perform the measurement $\{|+\rangle\langle+|,\ |-\rangle\langle-|\}$ on one-third of the systems.
   - Score $+1$ for each $|+\rangle\langle+|$ outcome
   - Score -1 for each $|-\rangle\langle-|$ outcome

   Expected value for each measurement: $\mathrm{Tr}(\sigma_x \rho)$
2. Perform the measurement $\{|+i\rangle\langle+i|,\ |-i\rangle\langle-i|\}$ on one-third of the systems.
   - Score $+1$ for each $|+i\rangle\langle+i|$ outcome
   - Score -1 for each $|-i\rangle\langle-i|$ outcome

   Expected value for each measurement: $\mathrm{Tr}(\sigma_y \rho)$
3. Perform the measurement $\{|0\rangle\langle0|,\ |1\rangle\langle1|\}$ on one-third of the systems.
   - Score $+1$ for each $|0\rangle\langle0|$ outcome
   - Score -1 for each $|1\rangle\langle1|$ outcome

   Expected value for each measurement: $\mathrm{Tr}(\sigma_z \rho)$

The density matrix ρ can now be approximated using this formula:

$$\rho = \frac{\mathbb{1} + \mathrm{Tr}(\sigma_x \rho)\sigma_x + \mathrm{Tr}(\sigma_y \rho)\sigma_y + \mathrm{Tr}(\sigma_z \rho)\sigma_z}{2}$$

Qubit tomography

We can alternatively perform tomography using the *tetrahedral measurement.*

$$|\phi_0\rangle = |0\rangle$$

$$|\phi_1\rangle = \frac{1}{\sqrt{3}}|0\rangle + \sqrt{\frac{2}{3}}|1\rangle$$

$$|\phi_2\rangle = \frac{1}{\sqrt{3}}|0\rangle + \sqrt{\frac{2}{3}}e^{2\pi i/3}|1\rangle$$

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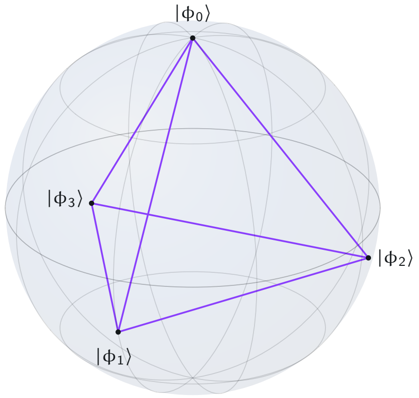

$$P_0 = \frac{|\phi_0\rangle\langle\phi_0|}{2} \qquad P_1 = \frac{|\phi_1\rangle\langle\phi_1|}{2} \qquad P_2 = \frac{|\phi_2\rangle\langle\phi_2|}{2} \qquad P_3 = \frac{|\phi_3\rangle\langle\phi_3|}{2}$$

Key formula:

$$\rho = \sum_{k=0}^{3}\left(3\,\mathrm{Tr}(P_k\rho) - \frac{1}{2}\right)|\phi_k\rangle\langle\phi_k|$$