Lecture 13

Variants of Turing machines

In this lecture we will continue to discuss the Turing machine model, focusing on ways in which the model can be changed without affecting its power.

13.1 Simple variants of Turing machines

There is nothing sacred about the specific definition of DTMs that we covered in the previous lecture. In fact, if you look at two different books on the theory of computation, you are pretty likely to see two definitions of Turing machines that differ in one or more respects.

For example, the definition we discussed specifies that a Turing machine's tape is infinite in both directions, but sometimes people choose to define the model so that the tape is only infinite to the right. Naturally, if there is a left-most tape square on the tape, the definition must clearly specify how the Turing machine is to behave if it tries to move its tape head left from this point. Perhaps the Turing machine immediately rejects if its tape head tries to move off the left edge of the tape, or the tape head might simply remain on the left-most tape square in this situation.

Another example concerns tape head movements. Our definition states that the tape head must move left or right at every step, while some alternative Turing machine definitions allow the possibility for the tape head to remain stationary. It is also common that Turing machines with multiple tapes are considered, and we will indeed consider this Turing machine variant shortly.

DTMs allowing stationary tape heads

Let us begin with a very simple Turing machine variant already mentioned above, where the tape head is permitted to remain stationary on a given step if the DTM designer wishes. This is an extremely minor change to the Turing machine definition, but because it is our first example of a Turing machine variant we will go through it in detail (perhaps more than it actually deserves).

If the tape head of a DTM is allowed to remain stationary, we would naturally expect that instead of the transition function taking the form

$$\delta : Q\backslash\{q_{\text{acc}}, q_{\text{rej}}\} \times \Gamma \rightarrow Q \times \Gamma \times \{\leftarrow, \rightarrow\}, \tag{13.1}$$

it would instead take the form

$$\delta : Q\backslash\{q_{\text{acc}}, q_{\text{rej}}\} \times \Gamma \rightarrow Q \times \Gamma \times \{\leftarrow, \downarrow, \rightarrow\}, \tag{13.2}$$

where the arrow pointing down indicates that the tape head does not move. Specifically, if it is the case that $\delta(p, a) = (q, b, \downarrow)$, then whenever the machine is in the state p and its tape head is positioned over a square that contains the symbol a, it overwrites a with b, changes state to q, and *leaves the position of the tape head unchanged*.

For the sake of clarity let us give this new model a different name, to distinguish it from the ordinary DTM model we already defined in the previous lecture. In particular, we will define a *stationary-head-DTM* to be a 7-tuple

$$M = (Q, \Sigma, \Gamma, \delta, q_0, q_{\text{acc}}, q_{\text{rej}}), \tag{13.3}$$

where each part of this tuple is just like an ordinary DTM except that the transition function δ takes the form (13.2).

Now, if we wanted to give a formal definition of what it means for a stationary-head-DTM to accept or reject, we could of course do that. This would require that we extend the yields relation defined in the previous lecture to account for the possibility that $\delta(p, a) = (q, b, \downarrow)$ for some choices of $p \in Q$ and $a \in \Gamma$. This is actually quite easy—we simply include the following third rule to the rules that define the yields relation for ordinary DTMs:

3. *Remaining stationary.* For every choice of $p \in Q\backslash\{q_{\text{acc}}, q_{\text{rej}}\}$, $q \in Q$, and $a, b \in \Gamma$ satisfying

   $$\delta(p, a) = (q, b, \downarrow), \tag{13.4}$$

   the yields relation includes these pairs for all $u \in \Gamma^*\backslash\{\sqcup\}\Gamma^*$ and $v \in \Gamma^*\backslash\Gamma^*\{\sqcup\}$:

   $$u(p, a)v \vdash_M u(q, b)v. \tag{13.5}$$

As suggested before, allowing the tape head to remain stationary does not actually change the computational power of the Turing machine model. The standard way to argue that this is so is through the technique of *simulation*. A standard DTM cannot leave its tape head stationary, so it cannot behave *precisely* like a stationary-head-DTM, but it is straightforward to *simulate* a stationary-head-DTM with an ordinary one—by simply moving the tape head to the left and back to the right (for instance), we can obtain the same outcome as we would have if the tape head had remained stationary. Naturally, this requires that we remember what state we are supposed to be in after moving left and back to the right, but it can be done without difficulty.

To be more precise, if

$$M = (Q, \Sigma, \Gamma, \delta, q_0, q_{\text{acc}}, q_{\text{rej}}) \tag{13.6}$$

is a stationary-head-DTM, then we can simulate this machine with an ordinary DTM

$$K = (R, \Sigma, \Gamma, \eta, q_0, q_{\text{acc}}, q_{\text{rej}}) \tag{13.7}$$

as follows:

1. For each state $q \in Q$ of M, the state set R of K will include q, as well as a distinct copy of this state that we will denote q'. The intuitive meaning of the state q' is that it indicates that K needs to move its tape head one square to the right and enter the state q.
2. The transition function η of K is defined as

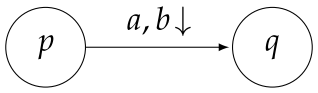
$$\eta(p,a) = \begin{cases} (q,b,\leftarrow) & \text{if } \delta(p,a) = (q,b,\leftarrow) \\ (q,b,\rightarrow) & \text{if } \delta(p,a) = (q,b,\rightarrow) \\ (q',b,\leftarrow) & \text{if } \delta(p,a) = (q,b,\downarrow) \end{cases} \tag{13.8}$$

   for each $p \in Q \backslash \{q_{\text{acc}}, q_{\text{rej}}\}$ and $a \in \Gamma$, as well as

$$\eta(q',c) = (q,c,\rightarrow) \tag{13.9}$$

   for each $q \in Q$ and $c \in \Gamma$.

Written in terms of state diagrams, one can describe this simulation as follows. Suppose that the state diagram of a stationary-head-DTM M contains a transition that looks like this:



The state diagram for K replaces this transition as follows:

$$\begin{tikzcd} p \arrow[r, "{a, b \leftarrow}"] & q' \arrow[r, "{c, c \rightarrow}"] & q \end{tikzcd}$$

Here, the transition from q' to q is to be included for every tape symbol $c \in \Gamma$. The same state q' can safely be used for every stationary tape head transition into q.

It is not hard to see that the computation of K will directly mimic the computation of M. The DTM K might take longer to run, because it sometimes requires two steps to simulate one step of M, but this does not concern us. The bottom line is that every language that is either decided or semidecided by a stationary-head-DTM is also decided or semidecided by an ordinary DTM.

The other direction is trivial: a stationary-head-DTM can easily simulate an ordinary DTM by simply not making use of its ability to leave the tape head stationary. Consequently, the two models are equivalent.

Thus, if you were to decide at some point that it would be more convenient to work with the stationary-head-DTM model, you could switch to this model—and by observing the equivalence we just proved, you would be able to conclude interesting facts concerning the original DTM model.

In reality, however, the stationary-head-DTM model just discussed is not a significant enough departure from the ordinary DTM model for us to be concerned with it—we went through this equivalence in detail only because it is a first example, and there will not likely be a need for us to refer specifically to the stationary-head-DTM model again.

DTMs with multi-track tapes

Another useful variant of the DTM model is one in which the tape has multiple tracks, as suggested by Figure 13.1. More specifically, we may suppose that the tape has k tracks for some positive integer k, and for each tape head position the tape has k separate tape squares that can each store a symbol. It is useful to allow the k different tracks to have possibly different tape alphabets $\Gamma_1, \ldots, \Gamma_k$. When the tape head scans a particular location on the tape, it can effectively see and modify all of the symbols stored on the tape tracks for this tape head location simultaneously.

For example, based on the picture in Figure 13.1 it appears as though the first tape track of this DTM stores symbols from the tape alphabet $\Gamma_1 = \{0, 1, \sqcup\}$, the second track stores symbols from the tape alphabet $\Gamma_2 = \{\#, \sqcup\}$, and the third track stores symbols from the tape alphabet $\Gamma_3 = \{\clubsuit, \heartsuit, \diamondsuit, \spadesuit, \sqcup\}$.

It turns out that this is not really even a variant of the DTM definition at all—it is just an ordinary DTM whose tape alphabet Γ is equal to the Cartesian product

$$\Gamma = \Gamma_1 \times \cdots \times \Gamma_k. \tag{13.10}$$

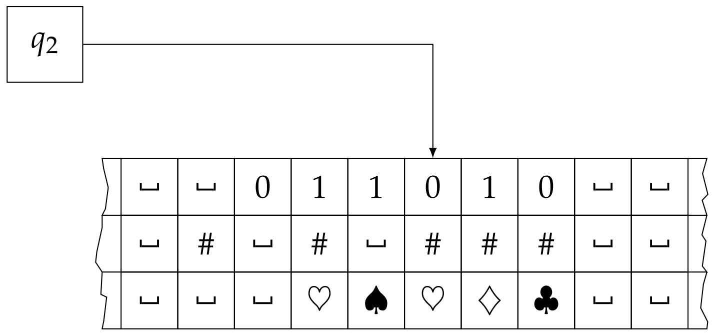


Figure 13.1: A DTM with a three-track tape.

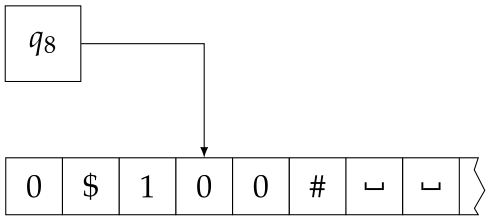


Figure 13.2: A DTM with a one-way infinite tape.

The tape alphabet of any DTM must include the input alphabet and a blank symbol, and so it should be understood that we identify each input alphabet symbol $\sigma \in \Sigma$ with the tape symbol $(\sigma, \sqcup, \ldots, \sqcup)$, and also that we consider the symbol $(\sqcup, \ldots, \sqcup)$ to be the blank symbol of the multi-track DTM.

DTMs with one-way infinite tapes

DTMs with one-way infinite tapes were mentioned before as a common alternative to DTMs with two-way infinite tapes. Figure 13.2 illustrates such a DTM. Let us say that if the DTM ever tries to move its tape head left when it is on the leftmost tape square, its head simply remains on this square and the computation continues—maybe it makes an unpleasant crunching sound in this situation.

It is easy to simulate a DTM with a one-way infinite tape using an ordinary DTM (with a two-way infinite tape). For instance, we could drop a special symbol, such as ✂, on the two-way infinite tape at the beginning of the computation, to the left of the input. The DTM with the two-way infinite tape will exactly mimic the behavior of the one-way infinite tape, but if the tape head ever scans the special ✂ symbol during the computation, it moves one square right without changing state.

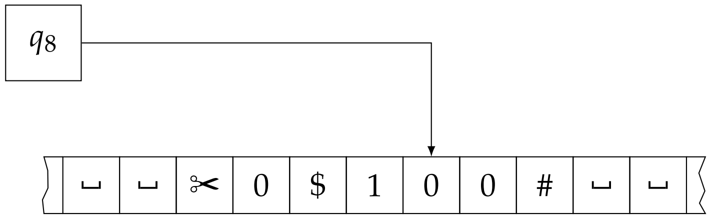


Figure 13.3: A DTM with a two-way infinite tape can easily simulate a DTM with a one-way infinite tape, like the one pictured in Figure 13.2, by writing a special symbol on the tape (in this case the symbol is ✂) that indicates where we should imagine the tape has been cut. When the tape head scans this symbol, the DTM adjusts its behavior accordingly.

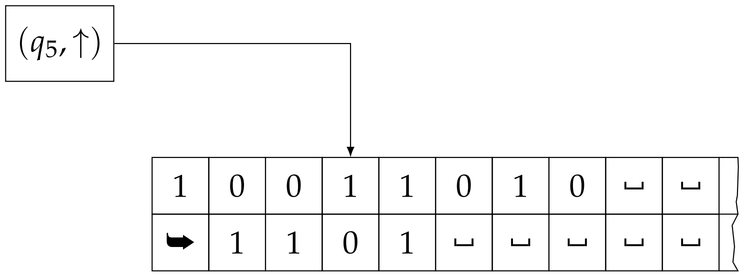


Figure 13.4: A DTM with a one-way infinite tape that simulates an ordinary DTM having a two-way infinite tape. The top track represents the portion of the two-way infinite tape that extends to the right and the bottom track represents the portion extending to the left.

This exactly mimics the behavior of the DTM with a one-way infinite tape that was suggested above.

Simulating an ordinary DTM having a two-way infinite tape with one having just a one-way infinite tape is slightly more challenging, but not difficult. Two natural ways to do it come to mind. The first way is suggested by Figure 13.4. In essence, the one-way infinite, two-track tape of the DTM suggested by the figure represents the tape of the original DTM being simulated, folded in half. The finite state control keeps track of the state of the DTM being simulated and which track of the tape stores the symbol being scanned. A special tape symbol, such as ➥, could be placed on the first square of the bottom track to assist in the simulation.

The second way to perform the simulation of a two-way infinite tape with a one-way infinite tape does not require two tracks, but will result in a simulation that is somewhat less efficient with respect to the number of steps required. A

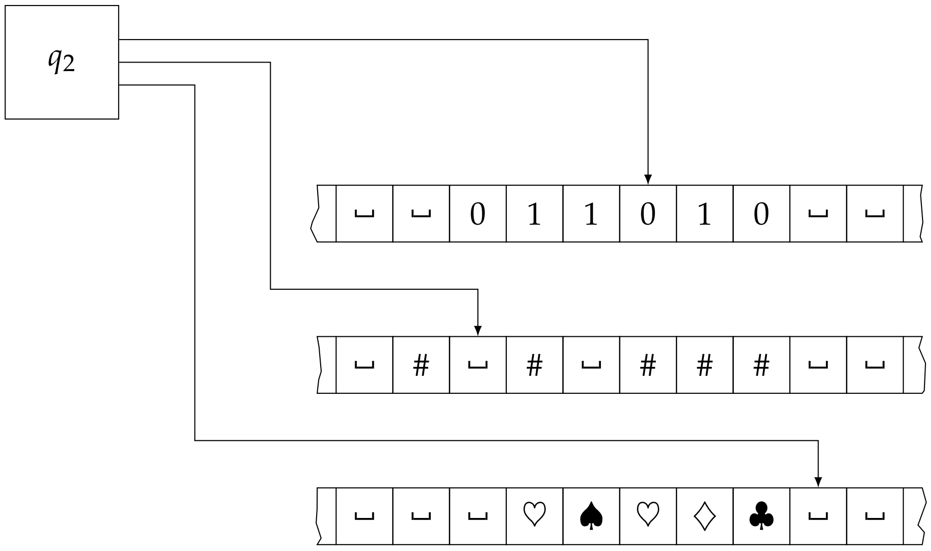


Figure 13.5: A DTM with three tapes.

special symbol could be placed in the left-most square of the one-way infinite tape, and anytime this symbol is scanned the DTM can transition into a subroutine in which every other symbol on the tape is shifted one square to the right in order to "make room" for a new square to the left. This would presumably require that we also use a special symbol marking the right-most non-blank symbol on the tape, so that the shifting subroutine can be completed—for otherwise we might not know when every (non-blank) symbol on the tape had been shifted one square to the right.

13.2 Multi-tape Turing machines

The last variant of the Turing machine model that we will consider is perhaps the most useful variant. A *multi-tape DTM* works in a similar way to an ordinary (single-tape) DTM, except that it has k tape heads that operate independently on k tapes, for some fixed positive integer k. For example, Figure 13.5 illustrates a multi-tape DTM with three tapes.

In general, a k-tape DTM is defined in a similar way to an ordinary DTM, except that the transition function has a slightly more complicated form. In particular, if the tape alphabets of a k-tape DTM are $\Gamma_1, \ldots, \Gamma_k$, then the transition function might take this form:

$$\delta : Q\backslash\{q_{\text{acc}}, q_{\text{rej}}\} \times \Gamma_1 \times \cdots \times \Gamma_k \to Q \times \Gamma_1 \times \cdots \times \Gamma_k \times \{\leftarrow, \downarrow, \rightarrow\}^k. \tag{13.11}$$

If we make the simplifying assumption that the same alphabet is used for each tape (which does not restrict the model, as we could always take this single tape alphabet to be the union $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_k$ of multiple alphabets), the transition function takes the form

$$\delta : Q \backslash \{q_{\text{acc}}, q_{\text{rej}}\} \times \Gamma^k \to Q \times \Gamma^k \times \{\leftarrow, \downarrow, \rightarrow\}^k. \tag{13.12}$$

(In both of these cases it is evident that the tape heads are allowed to remain stationary. Naturally you could also consider a variant in which every one of the tape heads must move at each step, but we may as well allow for stationary tape heads when considering the multi-tape DTM model—it is meant to be flexible and general, so as to make it easier to perform complex computations.) The interpretation of the transition function taking the form (13.12) is as follows. If it holds that

$$\delta(p, a_1, \ldots, a_k) = (q, b_1, \ldots, b_k, D_1, \ldots, D_k), \tag{13.13}$$

then if the DTM is in the state p and is reading the symbols $a_1, \ldots, a_k$ on its k tapes, then

1. the new state becomes q,
2. the symbols $b_1, \ldots, b_k$ are written onto the k tapes (overwriting the symbols $a_1, \ldots, a_k$), and
3. the j-th tape head either moves or remains stationary depending on the value $D_j \in \{\leftarrow, \downarrow, \rightarrow\}$, for each $j = 1, \ldots, k$.

One way to simulate a multi-tape DTM with a single-tape DTM is to store the contents of the k tapes, as well as the positions of the k tape heads, on separate tracks of a single-tape DTM whose tape has multiple tracks. For example, the configuration of the multi-tape DTM pictured in Figure 13.5 could be represented by a single-tape DTM as suggested by Figure 13.6.

Naturally, a simulation of this sort will require many steps of the single-tape DTM to simulate a single step of the multi-tape DTM. Let us refer to the multi-tape DTM as K and the single-tape DTM as M. To simulate one step of K, the DTM M needs many steps: it must first scan through the tape in order to determine which symbols are being scanned by the k tape heads of K, and store these symbols within its finite state control. Once it knows these symbols, it can decide what action K is supposed to take, and then implement this action—which means again scanning through the tape in order to update the symbols stored on the tracks that represent the tape contents of K and the positions of the tape heads, which may have to move. It would be complicated to write this all down carefully, and there are many specific ways in which this general idea could be carried out—but with enough time and motivation it would certainly be possible to give a formal definition for a single-tape DTM M that simulates a given multi-tape DTM K in this way.

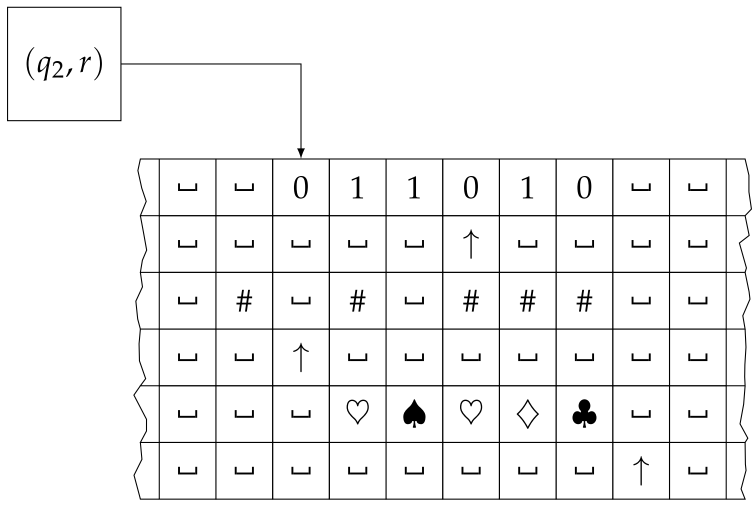


Figure 13.6: A single-tape DTM with a multi-track tape can simulate a multi-tape DTM. Here, the odd numbered tracks represent the contents of the tapes of the DTM illustrated in Figure 13.5, while the even numbered tracks store the locations of the tape heads of the multi-tape DTM. The finite state control of this single-tape DTM stores the state of the multi-tape DTM it simulates, but it would also need to store other information (represented by the component r in the picture) in order to carry out the simulation.